Geometry and Topology of random 2-complexes

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The Whitehead Conjecture

Let X be a 2-dimensional finite simplicial complex.

X is called *aspherical* if $\pi_2(X) = 0.$

Equivalently, X is aspherical if the universal cover $\tilde{X}$ is contractible.

Examples of aspherical 2-complexes: Σ_g with $g > 0$;

N_g with $g > 1$.

Non-aspherical are S^2 and P^2 (the real projective plane).

In 1941, J.H.C. Whitehead suggested the following question:

Is every subcomplex of an aspherical 2-complex also aspherical?

This question is known as the Whitehead conjecture.

Equivalently one may ask: suppose that K is a connected 2-complex with

$$\pi_2(K) \neq \mathbf{0}$$

and let

$$L = K \cup_f D^2, f : S^1 \to K$$

be obtained by attaching a 2-cell. Is

$$\pi_2(\mathrm{L}) \neq 0?$$

Theorem (J.F. Adams, 1955): If $L = K \cup_f D^2$ and $\pi_2(\mathrm{K}) \neq \mathbf{0}$, while $\pi_2(\mathrm{L}) = \mathbf{0}$, then the kernel of the homomorphism

$$\pi_1(K) \to \pi_1(L)$$

contains a nontrivial perfect subgroup.

This implies some (earlier) results of W.H. Cockcroft who considered the cases when $\pi_1(K)$ is finite, free, or free abelian.

Question :

Can one test the Whitehead Conjecture probabilistically?

Tasks :

1. Produce aspherical 2-complexes randomly;
2. Estimate the probability that the Whitehead Conjecture is satisfied

The Linial - Meshulam model

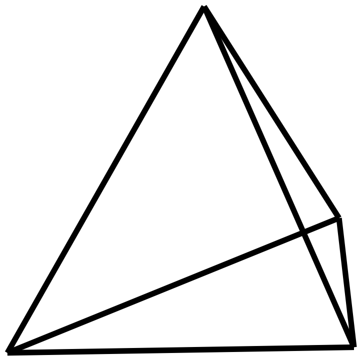
Consider the complete graph K_n on n vertices $\{1,2,\cdots,n\}$.
A random 2-complex X is obtained from K_n by adding each potential 2-simplex (ijk) at random, with probabiltiy $p \in (0,1)$, independently of each other. The finite probability space $Y(n,p)$ contains $2^{\binom{n}{3}}$ simplicial complexes satisfying

$$\Delta_n^{(1)} \subset Y \subset \Delta_n^{(2)}$$

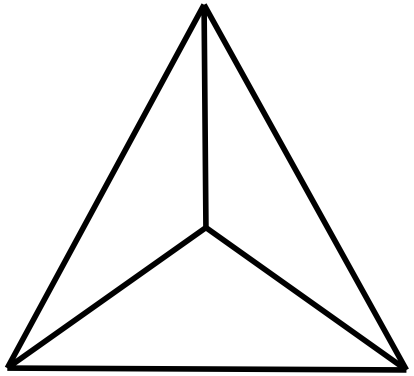
and the probability function $P : Y(n,p) \to R$ is given by

$$P(Y) = p^{f(Y)}(1-p)^{\binom{n}{3}-f(Y)}.$$

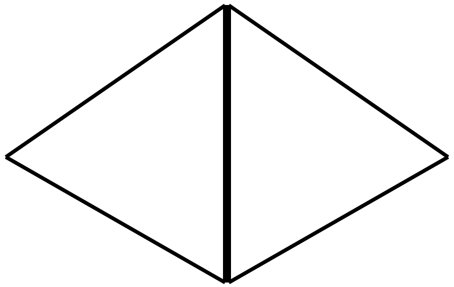
Case n=4:



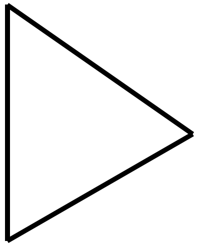
$$p^4$$



$$4p^3(1-p)$$



$$6p^2(1-p)^2$$

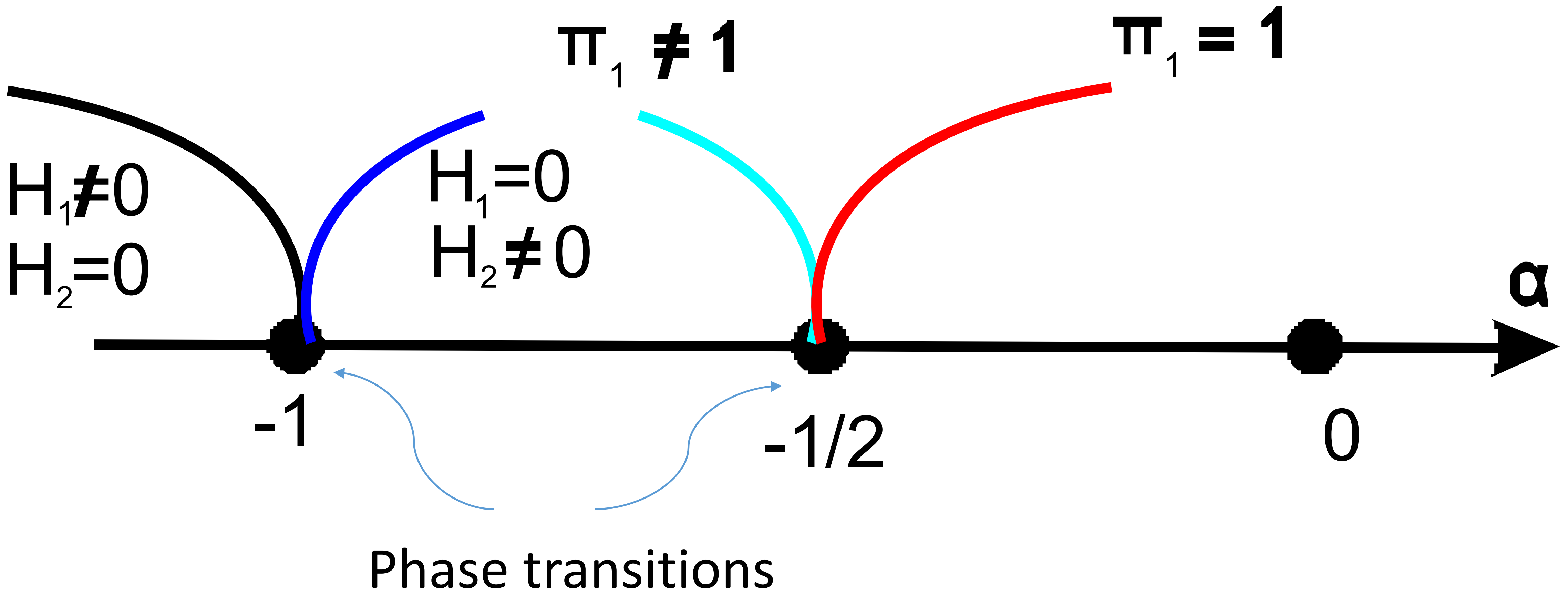


$$4p(1-p)^3$$

$$(1-p)^4$$

Topology of random 2 - complexes

For simplicity I will assume that $\boxed{p = n^{\alpha}}$, where $\alpha < 0$.


$\pi_1 \neq 1$
$\pi_1 = 1$
$H_1 \neq 0$
$H_2 = 0$
$H_1 = 0$
$H_2 \neq 0$
α
-1
-1/2
0
Phase transitions

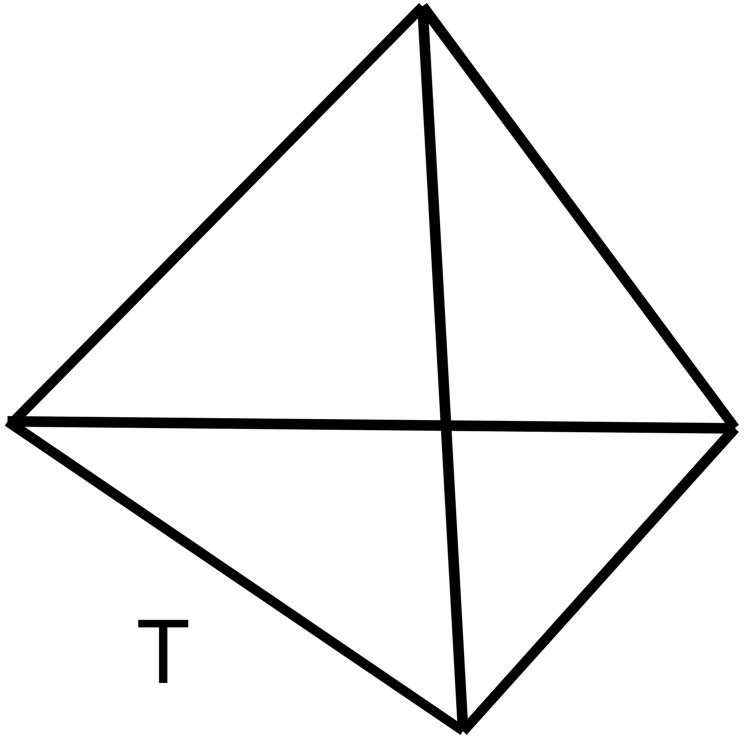
If $\alpha > -1$ then Y contains a subcomplex isomorphic to the tetrahedron T.

$g : V(T) \to \{1, 2, ..., n\}$

$J_g : Y(n,p) \to \mathbb{R}$

$J_g(Y) = \begin{cases} 1, & \text{if } g \text{ extends to an embedding } T \to Y \\ 0, & \text{otherwise} \end{cases}$

$E(J_g) = p^4$



$$X = \sum_g J_g, \quad X : Y(n,p) \to \mathbb{R}$$

X counts the number of tetrahedra in a random 2-complex.

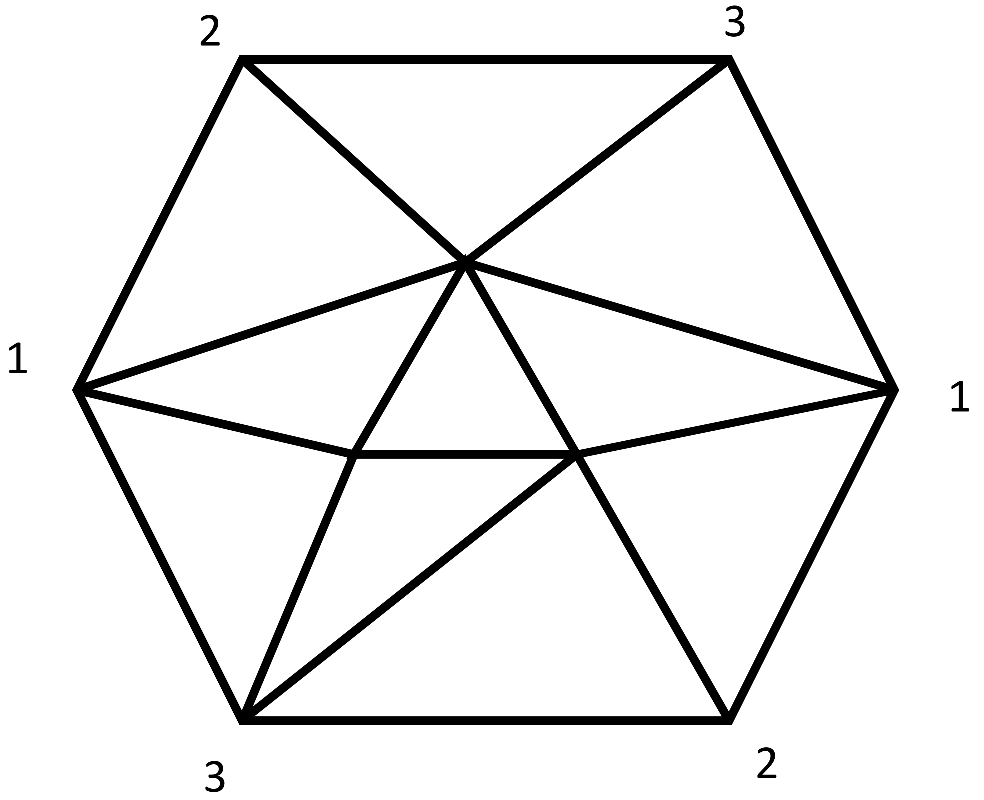
$$E(X) = \binom{n}{\mathbf{4}} p^{\mathbf{4}} \sim n^{\mathbf{4}} p^{\mathbf{4}} = n^{\mathbf{4}(\mathbf{1}+\alpha)} \to \infty,$$

if $\alpha > -\mathbf{1}$.

The results stated below were obtained jointly with Armindo Costa.

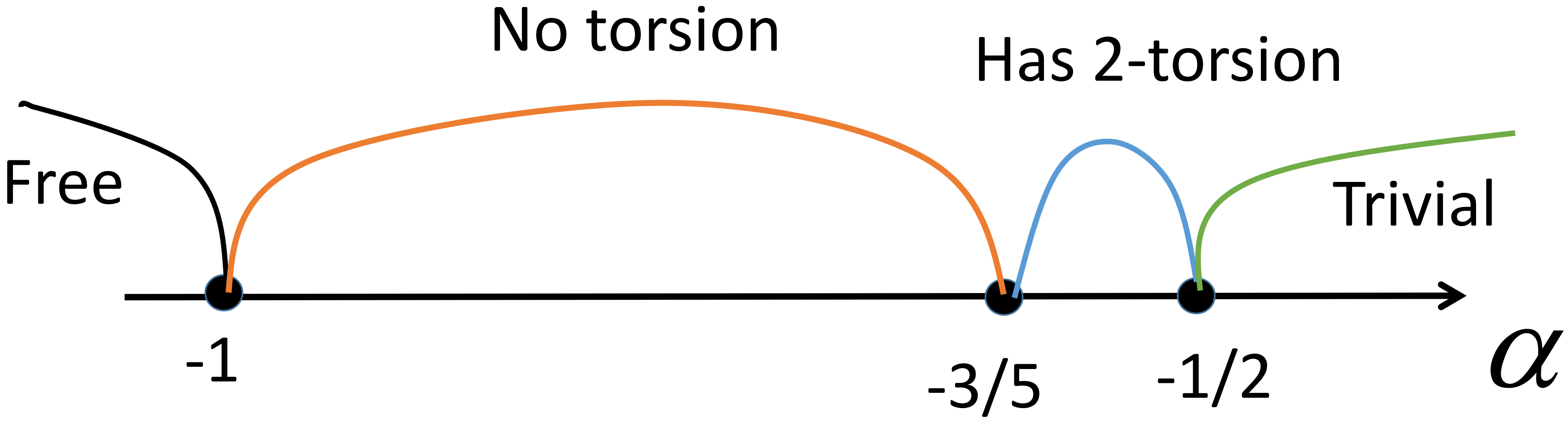
Theorem A**:**

Theorem B :

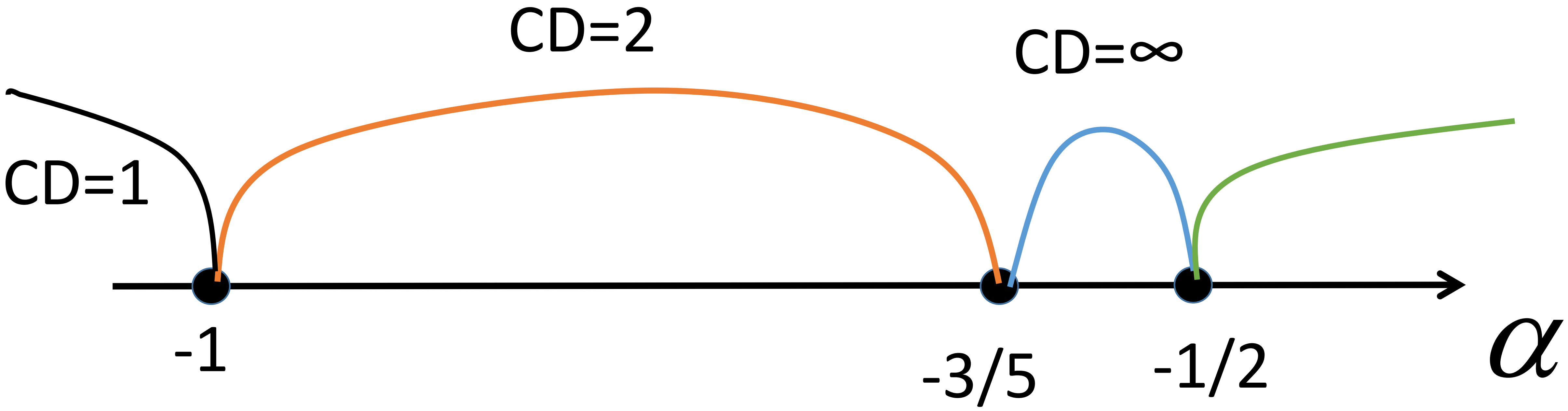


Triangulation of projective plane with 6 vertices and 10 faces.
(Note: 3/5=6/10)

Theorem C :

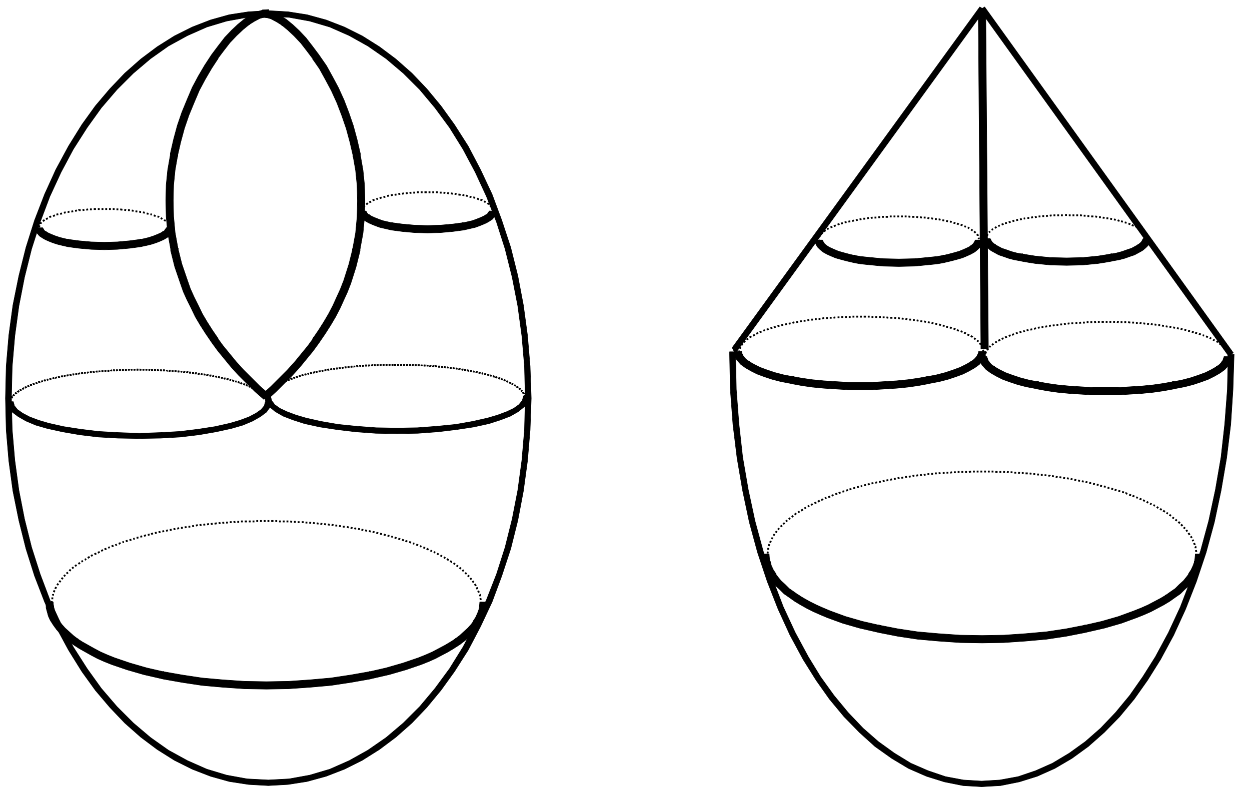


Torsion in the fundamental group of a random 2-complex



Cohomological dimension of fundamental group of a random 2-complex

Theorem D :



Complexes Z_2 (left) and Z_3 (right)

Corollary :

Assume that $\alpha < -1/2$.

Then a random 2-complex $Y \in Y(n, p)$ with probability tending to one has the following property: any aspherical subcomplex $Y' \subset Y$ satisfies the Whitehead Conjecture, i.e. every subcomplex $Y'' \subset Y'$ is also aspherical.

Isoperimetric constants

Let X be a simplicial 2-complex. For a simplicial null-homotopic loop $\gamma : S^1 \to X^{(1)}$ one defines the length $|\gamma|$ and the area $A_X(\gamma)$. The isoperimetric constant of X is defined as

$$I(X) = \inf\left\{\frac{|\gamma|}{A_X(\gamma)} ; \gamma : S^1 \to X\right\}.$$

$I(X) > 0$ iff the fundamental group $\pi_1(X)$ is hyperbolic.

The inequality $I(X) > a > 0$ *means that an* *isoperimetric inequality* $A_X(\gamma) < a^{-1} \cdot |\gamma|$ *is satisfied for any null-homotopic loop* $\gamma : S^1 \to X$.

Theorem (*Babson, Hoffman, Kahle,* **2011**) :
If the probability parameter α *satisfies* $\alpha < -1/2$ *then the fundamental group of a random 2-complex* $Y \in Y(n, p)$, $p = n^{\alpha}$, *is hyperbolic, a.a.s.*

Theorem :

If the probability parameter α satisfies $\alpha<-1/2$ then there exists a constant $C_\alpha > \mathbf{0}$, such that, with probability tending to one, a random 2-complex $Y \in Y(n, p)$, $p = n^\alpha$, has the following property: any subcomplex $Y' \subset Y$ satisfies $I(Y') > C_\alpha$.

Corollary :

For α<-1/2 a random 2-complex contains no subcomplexes homeomorphic to the torus T^2, a.a.s.

Minimal spheres

Let Y be a simplicial complex with $\pi_2(Y) \neq 0$.
Define $M(Y)$ as the minimal number of faces in a 2-complex Σ homeomorphic to 2-sphere such that there exists a homotopically nontrivial simplicial map $\Sigma \to Y$.
We also define $M(Y) = 0$ if $\pi_2(Y) = 0$.

Theorem :

If Y is a 2-complex satisfying $I\left(Y\right) > c > \mathbf{0}$ then

$$M\left(Y\right) \leq \left(\frac{\mathbf{16}}{c}\right)^{2}.$$

The proof of this deterministic statement uses an inequality of Papasoglu for Cheeger constants of triangulations of the sphere.

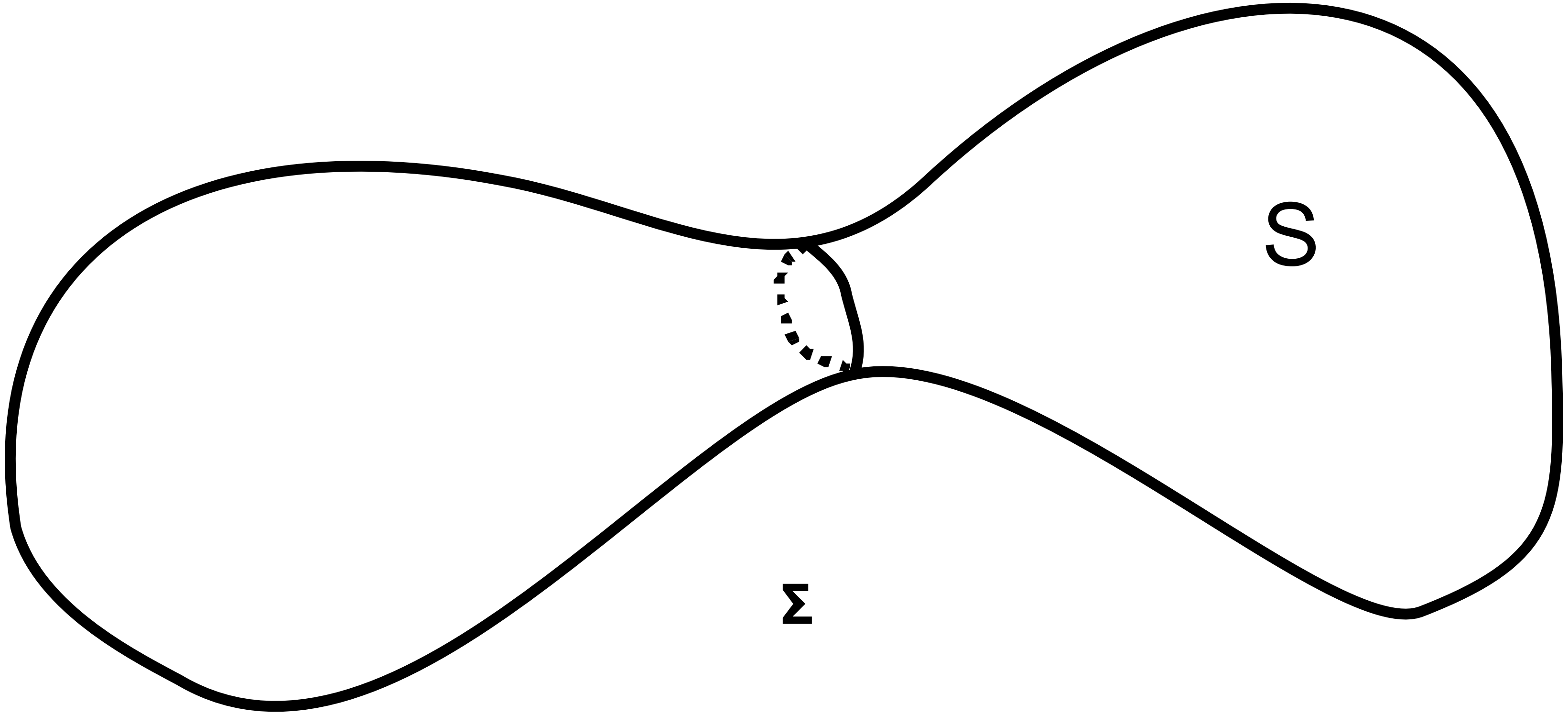
Proof :

Consider a homotopically nontrivial simplicial map $\Sigma \to Y$ *where* Σ *is homeomorphic to* S^2 *and* $A(\Sigma) = M(Y)$.

Consider the Cheeger constant of Σ,

$$h(\Sigma) = \min_{S \subset \Sigma} \left\{ \frac{|\partial S|}{A(S)} ; A(S) \leq A(\Sigma)/2 \right\}.$$

Here S *is a subcomplex homeomorphic to the disc.*


S
Σ

One can show that $I(Y) > c > 0$ implies $h(\Sigma) \geq c$.

On the other hand, Papasoglu proved an inequality

$$h(\Sigma) \leq \frac{16}{\sqrt{A(\Sigma)}}.$$

Combining we obtain

$$M(Y) = A(\Sigma) \leq \left(\frac{16}{h(\Sigma)}\right)^2 \leq \left(\frac{16}{c}\right)^2.$$

Theorem :

If the probability parameter α satisfies $\alpha < -1/2$ then for some constant $C_\alpha > 0$ a random 2-complex $Y \in Y(n, p)$, $p = n^\alpha$, with probability tending to one has the following property: for any subcomplex $Y' \subset Y$ one has $M(Y') \leq C_\alpha$.

Gromov's local – to – global Principle

Theorem :

Let X be a finite 2-complex and let $C > \mathbf{0}$ be a constant such that any pure subcomplex $S \subset X$ having at most $\mathbf{44}^{\mathbf{3}} \cdot C$ faces satisfies $I(S) \geq C$.

Then $I(X) \geq C \cdot \mathbf{44}^{\mathbf{-1}}$.

Classification of minimal cycles

A finite 2-complex Z is said to be a minimal cycle if $b_2(Z) = 1$ and for any proper subcomplex $Z' \subset Z$ one has $b_2(Z') = 0$.

For a minimal cycle Z we denote

$$\boxed{\mu(Z) = \frac{v(Z)}{f(Z)} \in \mathbb{Q}.}$$

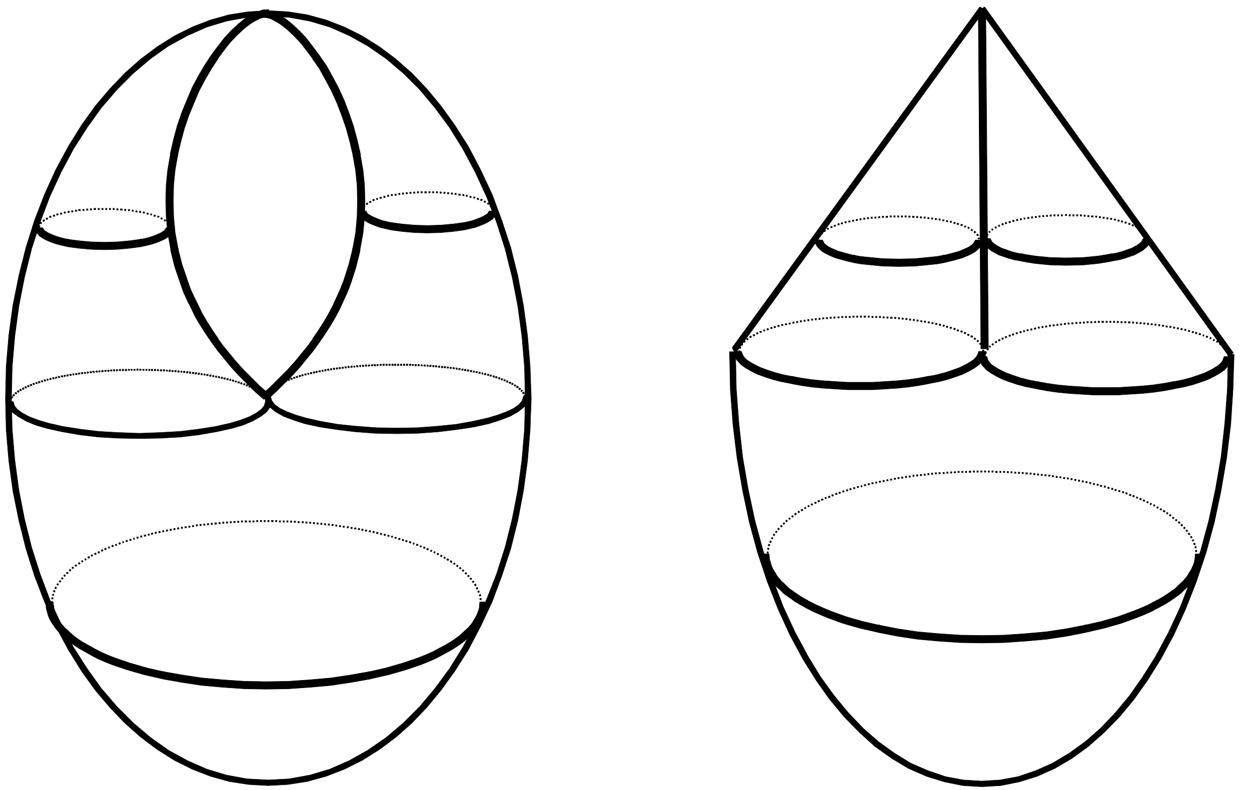
We are interested in describing all minimal cycles satisfying $\mu(Z) > 1/2$.

Theorem MinCycle :

Any minimal cycle Z satisfying $\mu(Z) > 1/2$ is homeomorphic to one of four complexes

$Z_1 = S^2$, Z_2, Z_3, $Z_4 = P^2 \cup \Delta^2$,

where $P^2 \cap \Delta^2 = P^1$ and Z_2, Z_3 are shown on the following slide.



Complexes Z_2 (left) and Z_3 (right)

Theorem :

A random 2-complex $Y \in Y(n, p)$, $p = n^{\alpha}$, $\alpha < -1/2$ with probability tending to one has the following property: for a subcomplex $Y' \subset Y$ the following properties are equivalent:

(A) Y' is aspherical;

(B) Y' contains no subcomplexes with at most $-4(1+2\alpha)^{-1}$ faces which are homeomorphic to S^2, P^2, Z_2, Z_3.

Corollary :

Assume that $\alpha < -1/2$.

Then a random 2-complex $Y \in Y(n,p)$ *with probability tending to one has the following property: any aspherical subcomplex* $Y' \subset Y$ *satisfies the Whitehead Conjecture, i.e. every subcomplex* $Y'' \subset Y'$ *is also aspherical.*

Proof of $(B) \Rightarrow (A)$

Let $Y' \subset Y$, $Y \in Y(n, p)$, $p = n^{\alpha}$.

Then $M(Y') < C_{\alpha}$, a.a.s.

There are finitely many isomorphism types of triangulations $\{S_j\}$ of the 2-sphere with at most C_{α} faces.

There are also finitely many simplicial quotients $\{\varphi_j(S_j)\}$ of such triangulations.

The quotients satisfying $\mu(\phi_j(S_j)) < -\alpha$ cannot be embedded into Y.

Thus we shall only consider the quotients satisfying $\mu(\phi_j(S_j)) \geq -\alpha > \mathbf{1/2}$.

Case when $b_2\left(\phi_j\left(S_j\right)\right) > 0$.

Then the image $\phi_j\left(S_j\right)$ contains a minimal cycle which by Theorem MinCycle is homeomorphic to one of Z_1, Z_2, Z_3, Z_4.

$\mu\left(Z_j\right) \geq -\alpha$ implies $f\left(Z_j\right) \leq -4\left(1+2\alpha\right)^{-1}$ which contradicts our assumption $\left(A\right)$.

Case $b_2\left(\phi_j\left(S_j\right)\right) = 0$.

Then one shows that the image $\phi_j\left(S_j\right)$

contains a projective plane with at most

$-4\left(1+2\alpha\right)^{-1}$ faces which contradicts our assumption $\left(A\right)$.

What does all this mean for the deterministic Whitehead Conjecture?

Theorem C :

Let $m \geq 3$ be an odd prime.
If the proability parameter α satisfies
$\alpha < -1/2$ then, with probability tending to one as
$n \to \infty$, a random 2-complex $Y \in Y(n, p)$ has the following
property: the fundamental group of any subcomplex
$Y' \subset Y$ has no m – torsion.

Sketch of the proof

Consider the Moore surface $M\left(\mathbb{Z}_m, \mathbf{1}\right) = S^1 \cup_m D^2$,

$\pi_1\left(M\left(\mathbb{Z}_m, \mathbf{1}\right)\right) = \mathbb{Z}_m$.

Maps $M\left(\mathbb{Z}_m, \mathbf{1}\right) \to Y$ inducing mono on π_1 describe m – torison in $\pi_1\left(Y\right)$.

Σ -triangulation of the Moore surface.

We shall consider simplicial maps $\Sigma \to Y$ such that:

- they induce mono on π_1
- have shortest possible length of the singular curve $C \subset \Sigma$
- have smallest possible area (the number of faces)

One defines the number $N_m(Y)$ as the number of faces in Σ above.

Lemma :

If $I\left(Y\right) > c > \mathbf{0}$ then

$$N_m\left(Y\right) \leq \left(\frac{\mathbf{6}m}{c}\right)^{\mathbf{2}}$$

The proof uses systollic inequality

$$sys\left(\Sigma\right) \leq \mathbf{6} \bullet A\left(\Sigma\right)^{\mathbf{1}/\mathbf{2}}$$

(Gromov, Katz, Rudyak,$\cdots$)