Syzygies and Multi-Dimensional Persistence

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Set-up

Begin with a simplicial complex X with a filtration

$$X_\bullet = \{X_v | v \in \mathbb{N}^n\}.$$

Here, $\mathbb{N}^n$ is ordered via $u = (u_1, \dots, u_n) \le v = (v_1, \dots, v_n)$ if $u_i \le v_i$ for $i = 1, \dots, n$.

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Persistence in this context is fuzzy–"barcodes" do not really exist when $n > 1$. In fact, persistence modules

$$M = \bigoplus_{v \in \mathbb{N}^n} H_i(X_v; k)$$

are parametrized by a certain quotient $G \backslash V$, where V is the algebraic variety of A_n-modules with generators and relations:

$$\begin{aligned} \xi_0 &= \text{multiset in } \mathbb{N}^n \text{ giving locations where homology classes are born} \\ \xi_1 &= \text{multiset in } \mathbb{N}^n \text{ giving locations where homology classes die} \end{aligned}$$

and G is a certain algebraic group ($A_n = k[x_1, \ldots, x_n]$).

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Note the following:

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If $n = 1$, there are no higher Tor groups and that is why one-dimensional persistence is so neat.

But if $n \geq 2$, we have higher Tor groups.

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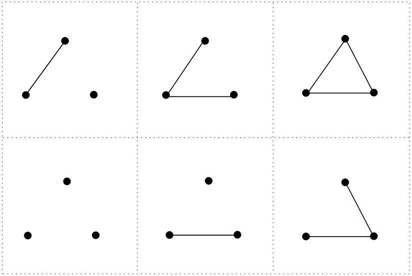
What can we say about these ξ_i? Do they have some geometric meaning?

A Simple Example



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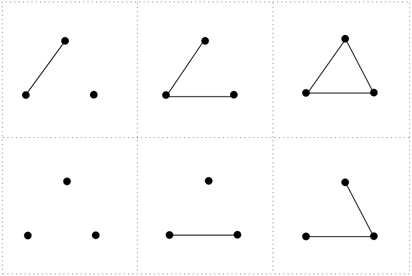

A filtration of the circle

$$\begin{array}{ccc} k^2 & k & k \\ k^3 & k^2 & k \end{array} \qquad \begin{array}{ccc} 0 & 0 & k \\ 0 & 0 & 0 \end{array}$$

$$H_0 \qquad\qquad H_1$$

The modules H_0 and H_1

For these modules, we have the following sets:

$$
\begin{aligned}
\xi_0(H_0) &= \{((0,0),3)\} \\
\xi_1(H_0) &= \{((0,1),1),((1,0),1),((2,0),1)\} \\
\xi_2(H_0) &= \{((2,1),1)\}
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Note the relationship between $\xi_2(H_0)$ and $\xi_0(H_1)$.

Hypertor

There is a functorial way to analyze this relationship. Consider the chain complex

$$C_\bullet(X_\bullet) = \{\cdots \to C_i(X_\bullet) \xrightarrow{\partial} C_{i-1}(X_\bullet) \cdots\}$$

This is a chain complex in the category of A_n-modules and we may consider the hypertor modules

$$\mathbf{Tor}_p^{A_n}(C_\bullet(X_\bullet), M)$$

for any module M. Here, we will consider only $M = k$ sitting in degree $(0, 0, \ldots, 0)$.

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As usual, there are two spectral sequences for computing this. Taking horizontal homology first:

$$E^2_{pq} = \mathrm{Tor}_p^{A_n}(H_q(X_\bullet), k) \Rightarrow \mathbf{Tor}_{p+q}^{A_n}(C_\bullet(X_\bullet), k)$$

Note: We have a map

$$d^2_{2,q} : \mathrm{Tor}_2^{A_n}(H_q(X_\bullet), k) \to \mathrm{Tor}_0^{A_n}(H_{q+1}(X_\bullet), k)$$

That is, we have a functorial way to relate elements of $\xi_2(H_q(X_\bullet))$ to $\xi_0(H_{q+1}(X_\bullet))$.

The Circle, Again

In the case of the simple circle example shown earlier, we have one such interesting map:

$$d^2_{2,0} : \mathrm{Tor}_2^{A_2}(H_0(X_\bullet), k) \to \mathrm{Tor}_0^{A_2}(H_1(X_\bullet), k).$$

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By choosing suitable resolutions of H_0 and H_1 (e.g., H_1 is a free A_2-module with a single generator in degree $(2, 1)$), we see that

$$\begin{aligned} \mathrm{Tor}_2^{A_2}(H_0(X_\bullet), k) &= k(2, 1) \\ \mathrm{Tor}_0^{A_2}(H_1(X_\bullet), k) &= k(2, 1) \end{aligned}$$

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So the function in question is a map $k(2,1) \to k(2,1)$.

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- Note that $C_0(X_\bullet)$ and $C_1(X_\bullet)$ are free A_2-modules. Thus,

$$\mathbf{Tor}_i^{A_2}(C_\bullet(X_\bullet), k) = H_i(C_\bullet(X_\bullet) \otimes_{A_2} k).$$

It is easy to see that

$$\mathbf{Tor}_0 = k(0,0)^3$$

and

$$\mathbf{Tor}_1 = k(0,1) \oplus k(1,0) \oplus k(2,0)$$

living in degrees $(0,0)$ and $(1,0)$, respectively, in the E^2-term of the spectral sequence. Since $E^3 = E^\infty$ and $\mathbf{Tor}_2 = 0$, we must have that $d^2_{2,0} : k(2,1) \to k(2,1)$ is an isomorphism.

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- Do the dirty work–choose a Cartan-Eilenberg resolution of $C_\bullet(X_\bullet)$ and compute directly that $d^2_{2,0} = -\mathrm{id}$.

Geometric Interpretation

Observe that in our circle example, the generator of $\mathrm{Tor}_2^{A_2}(H_0(X_\bullet), k)$ represents the first location where a collection of relations in H_0 is not an independent set. This happens either because there is a duplication of relations, or, as in this case, because they have come together to form a 1-cycle.

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This is a general result.

Theorem. The kernel of the map

$$d^2_{2,q} : \mathrm{Tor}_2^{A_n}(H_q(X_\bullet), k) \to \mathrm{Tor}_0^{A_n}(H_{q+1}(X_\bullet), k)$$

is generated by syzygies resulting from the same relation being imposed in $H_q(X_\bullet)$ in multiple degrees. If a nonzero $w \in \mathrm{Tor}_0^{A_n}(H_{q+1}(X_\bullet), k)$ is in the image of $d^2_{2,q}$, say $d^2_{2,q} z = w$, then $w = \sum \alpha_i w_i$ for some $(q+1)$-simplices w_i where each w_i corresponds to an element of $\mathrm{Tor}_1^{A_n}(B_q(X_\bullet), k)$ and z gives a syzygy among the w_i.

The Circle, One More Time

Denote the generators of $C_1(X_\bullet, k)$ by a, b, c, sitting in degrees $(0,1)$, $(1,0)$, and $(2,0)$, respectively. For simplicity, take $k = \mathbb{F}_2$. Then the generator of $\mathrm{Tor}_0^{A_2}(H_1(X_\bullet), k)$ is represented by the cycle $w = a + b + c$. Write $\partial a = x + z$, $\partial b = x + y$, and $\partial c = y + z$; these belong to $B_0(X_\bullet)$. Then we have the syzygy

$$z = x_1^2(x+z) + x_1x_2(x+y) + x_2(y+z) \in \mathrm{Tor}_1^{A_2}(B_0(X_\bullet), k)$$

and $d_{2,0}^2 z = w$.

When $n = 2$, this is the whole story. For $n > 2$, we get higher differentials

$$d^{\ell}_{\ell,q} : E^{\ell}_{\ell,q} \to E^{\ell}_{0,q+\ell-1}.$$

These relate elements of $\mathrm{Tor}^{A_n}_{\ell}(H_q(X_\bullet), k)$ to elements of $\mathrm{Tor}^{A_n}_{0}(H_{q+\ell-1}(X_\bullet), k)$.

Question. Is there an interesting geometric interpretation of these maps?

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