Random Latin Squares and 2-dimensional Expanders

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joint work with Alex Lubotzky

Applied and Computational Algebraic Topology
Bremen, July 2013

Plan

Expansion in Graphs and Complexes

- Expander Graphs
- Cohomological Expansion of Complexes
- The Topological Overlap Property

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Latin Square Complexes

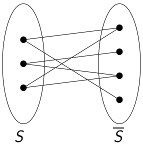
- A Model for Random 2-Complexes
- Spectral Gaps and 2-Expansion
- Large Deviations for Latin Squares
- Random LS-Complexes are 2-Expanders
- Related Questions and Open Problems

The Graphical Cheeger Constant

Edge Cuts

For a graph $G = (V, E)$ and $S \subset V$, $\overline{S} = V - S$ let

$$e(S, \overline{S}) = |\{e \in E : |e \cap S| = 1\}|.$$



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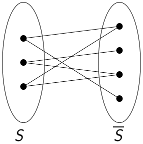

Cheeger Constant

$$h(G) = \min_{0<|S|\leq\frac{|V|}{2}} \frac{e(S, \overline{S})}{|S|}.$$

Expander Graphs

(d, ϵ)-Expanders

A family of graphs $\{G_n = (V_n, E_n)\}_n$ with $|V_n| \to \infty$ with two seemingly contradicting properties:

- High Connectivity: $h(G_n) \geq \epsilon$.
- Sparsity: $\max_v \deg_{G_n}(v) \leq d$.

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Lubotzky-Phillips-Sarnak, Margulis:

Ramanujan Graphs - an "optimal" family of expanders.

Spectral Gap

Laplacian Matrix

$G = (V, E)$ a graph, $|V| = n$.

The Laplacian of G is the $V \times V$ matrix L_G:

$$\mathrm{L_G(u,v)} = \begin{cases} \deg(u) & u = v \\ -1 & uv \in E \\ 0 & \text{otherwise.} \end{cases}$$

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Eigenvalues of L_G

$$0 = \mu_1(G) \leq \mu_2(G) \leq \cdots \leq \mu_n(G).$$

$$\mu_2(G) = \text{Spectral Gap of } G.$$

Expansion and Spectral Gap

Theorem (Alon-Milman, Tanner):

For all $\emptyset \neq S \subsetneqq V$

$$e(S, \overline{S}) \geq \frac{|S||\overline{S}|}{n} \mu_2.$$

In particular

$$h(G) \geq \frac{\mu_2}{2}.$$

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Expanders can thus be defined using the spectral gap.

Why Expanding Graphs?

Uses of Expanders

- Construction of efficient communication networks.
- Randomization reduction in probabilistic algorithms.
- Construction of good error correcting (LDPC) codes.
- Tools in computational complexity lower bounds.

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Interactions with Other Areas

- Expansion and Kazhdan's property T.
- Expanders as spaces of maximal Euclidean distortion.
- Dimension expanders and representation theory.
- Expanders on finite simple groups.

What are Expanding Complexes?

Three Notions of Expansion

- Combinatorial: via the mixing property.
- Spectral: via eigenvalues of the higher Laplacians.
- Cohomological: via the Hamming weights of coboundaries.

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Cohomological Expansion

This notion is strongly tied to topology, e.g. :

- Linial-M-Wallach: Homology of random complexes.
- Gromov: The topological overlap property.
- Gundert-Wagner: Laplacians of random complexes.
- Dotterrer-Kahle: Expansion of random subcomplexes.

Simplicial Cohomology

X a simplicial complex on V, R a fixed abelian group.
i-face of $\sigma = [v_0, \cdots, v_k]$ is $\sigma_i = [v_0, \cdots, \widehat{v_i}, \cdots, v_k]$.
$C^k(X) = k$-cochains $=$ skew-symmetric maps $\phi : X(k) \to R$.
Coboundary Operator $d_k : C^k(X) \to C^{k+1}(X)$ given by

$$d_k\phi(\sigma) = \sum_{i=0}^{k+1} (-1)^i \phi(\sigma_i) \quad .$$

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$d_{-1} : C^{-1}(X) = R \to C^0(X)$ given by
$d_{-1}a(v) = a$ for $a \in R$, $v \in V$.
$Z^k(X) =$ k-cocycles $= \ker(d_k)$.
$B^k(X) =$ k-coboundaries $= Im(d_{k-1})$.
k-th reduced cohomology group of X:

$$\tilde{\mathrm{H}}^k(X) = \tilde{\mathrm{H}}^k(X;R) = Z^k(X)/B^k(X) \quad .$$

Cut of a Cochain

Cut determined by a k-cochain $\phi \in C^k(X;R)$:

$$supp(d_k\phi) = \{\tau \in X(k+1) \;:\; d_k\phi(\tau) \neq 0\} \ .$$

Cut Size of ϕ: $\|d_k\phi\| = |supp(d_k\phi)|$.

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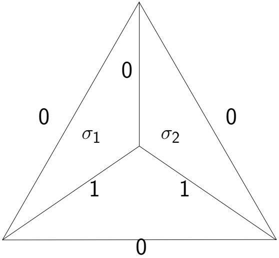

$$\|d_1\phi\| = |\{\sigma_1, \sigma_2\}| = 2$$

Hamming Weight of a Cochain

The Weight of a k-cochain $\phi \in C^k(X;R)$:

$$\|[\phi]\| = \min \ \{ \ |supp(\phi + d_{k-1}\psi)| \ : \ \psi \in C^{k-1}(X;R) \ \}.$$

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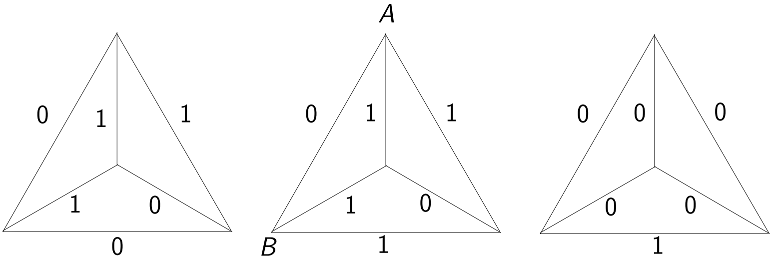
Example: $\|\phi\| = 3$ but $\|[\phi]\| = 1$



ϕ $\quad$ $d_0 1_{A,B}$ $\quad$ $\phi - d_0 1_{A,B}$

Expansion of a Complex

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Remarks:

- $h_k(X;R) > 0 \Leftrightarrow \tilde{H}^k(X;R) = 0$.
- In the sequel: $h_k(X) = h_k(X;\mathbb{F}_2)$.

Cheeger Constants of a Simplex

Δ_{n-1} = the $(n-1)$-dimensional simplex on $V = [n]$.

Claim [M-Wallach, Gromov]:

$$h_{k-1}(\Delta_{n-1}) = \frac{n}{k+1}.$$

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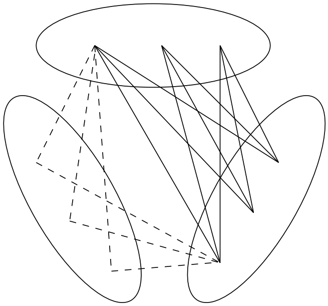
Example:

$[n] = \cup_{i=0}^{k} V_i \ , \ |V_i| = \frac{n}{k+1}$

$\phi = 1_{V_0 \times \cdots \times V_{k-1}}$

$\|[\phi]\| = (\frac{n}{k+1})^k$

$\|d_{k-1}\phi\| = (\frac{n}{k+1})^{k+1}$



The Affine Overlap Property

Number of Intersecting Simplices

For $A = \{a_1, \ldots, a_n\} \subset \mathbb{R}^k$ and $p \in \mathbb{R}^k$ let

$$\gamma_A(p) = |\{\sigma \subset [n] : |\sigma| = k + 1 \ , \ p \in \mathrm{conv}\{a_i\}_{i \in \sigma}\}|.$$

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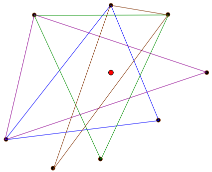
Theorem [Bárány]:

There exists a $p \in \mathbb{R}^k$ such that

$$f_A(p) \geq \frac{1}{(k+1)^k} \binom{n}{k+1} - O(n^k).$$



The Topological Overlap Property

Number of Intersecting Images

For a continuous map $f : \Delta_{n-1} \to \mathbb{R}^k$ and $p \in \mathbb{R}^k$ let

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Theorem [Gromov]:

There exists a $p \in \mathbb{R}^k$ such that

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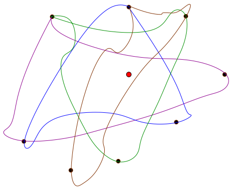

Topological Overlap and Expansion

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Expansion Condition on X

Suppose that for all $0 \leq i \leq k-1$

$$h_i(X) \geq \epsilon \cdot \frac{f_{i+1}(X)}{f_i(X)}.$$

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Theorem [Gromov]

There exists a $\delta = \delta(k, \epsilon)$ such that for any continuous map $f : X \to \mathbb{R}^k$ there exists a $p \in \mathbb{R}^k$ such that

$$\gamma_f(p) \geq \delta f_k(X).$$

Expander Complexes

Degree of a Simplex

For $\sigma \in X(k-1)$ let $\deg(\sigma) = |\{\tau \in X(k) : \sigma \subset \tau\}|$.
$D_{k-1}(X) = \max_{\sigma \in X(k-1)} \deg(\sigma)$.

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(k, d, ϵ)-Expanders

A family of Complexes $\{X_n\}_n$ with $f_0(X_n) \to \infty$ such that

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Random Complexes as Expanders

$Y \in Y_k(n, p = \frac{k^2 \log n}{n})$ is a.a.s. a $(k, \log n, 1)$-expander.

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Problem

Do there exist (k, d, ϵ)-expanders with $k \geq 2$ and fixed d, ϵ ?

Latin Squares

Definitions

$\mathbb{S}_n$ = Symmetric group on $[n]$.
$(\pi_1, \ldots, \pi_k) \in \mathbb{S}_n^k$ is legal if $\pi_i(\ell) \neq \pi_j(\ell)$ for all ℓ and $i \neq j$.
A Latin Square is a legal n-tuple $L = (\pi_1, \ldots, \pi_n) \in \mathbb{S}_n^n$.
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The Usual Picture

$L = (\pi_1, \ldots, \pi_n) \leftrightarrow T_L \in M_{n \times n}([n])$
$T_L(i, \pi_k(i)) = k$ for $1 \leq i, k \leq n$.

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Example for $n = 4$

$\pi = (1234)$

$L = (Id, \pi, \pi^2, \pi^3)$

$T_L =$

1	2	3	4
4	1	2	3
3	4	1	2
2	3	4	1

The Complete 3-Partite Complex

$$V_1 = \{a_1, \ldots, a_n\}\ ,\ \ V_2 = \{b_1, \ldots, b_n\}\ ,\ \ V_3 = \{c_1, \ldots, c_n\}$$

$$T_n = V_1 * V_2 * V_3 = \{\sigma \subset V : |\sigma \cap V_i| \leq 1 \text{ for } 1 \leq i \leq 3\}$$

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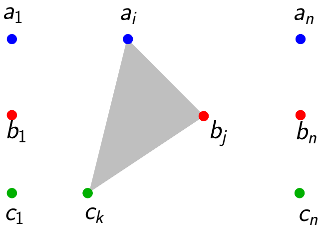

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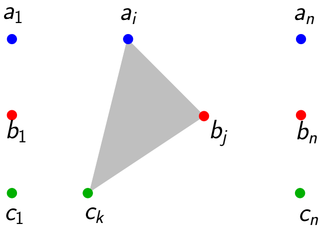

$$T_n \simeq S^2 \vee \cdots \vee S^2 \qquad (n-1)^3 \ \text{ times}$$

Latin Square Complexes

$L = (\pi_1, \ldots, \pi_n) \in \mathcal{L}_n$ defines a complex $Y(L) \subset T_n$ by

$$Y(L)(2) = \left\{ [a_i, b_j, c_{\pi_i(j)}] \quad : \quad 1 \le i, j \le n \right\}.$$

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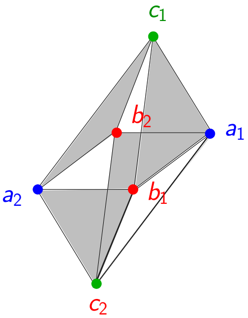
Example: $n = 2$

$$L = \begin{array}{|c|c|} \hline 1 & 2 \\ \hline 2 & 1 \\ \hline \end{array}$$

$Y(L) =$ 

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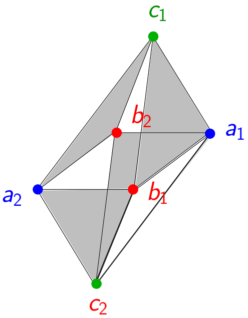

$$Y\left(\begin{array}{|c|c|} \hline 1 & 2 \\ \hline 2 & 1 \\ \hline \end{array} \right) \cup Y\left(\begin{array}{|c|c|} \hline 2 & 1 \\ \hline 1 & 2 \\ \hline \end{array} \right) = T_2$$

Random Latin Squares Complexes

Multiple Latin Squares

For $\underline{L}^d = (L_1, \ldots, L_d) \in \mathcal{L}_n^d$ let $Y(\underline{L}^d) = \cup_{i=1}^d Y(L_i)$.

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The Probability Space $\mathcal{Y}(n, d)$

$\mathcal{L}_n^d = d$-tuples of Latin squares of order n with uniform measure.
$\mathcal{Y}(n, d) = \{Y(\underline{L}^d) : \underline{L}^d \in \mathcal{L}_n^d\}$ with induced measure from $\mathcal{L}_n^d$.

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The Probability Space $\mathcal{Y}(n, d)$

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$\mathcal{Y}(n, d) = \{Y(\underline{L}^d) : \underline{L}^d \in \mathcal{L}_n^d\}$ with induced measure from $\mathcal{L}_n^d$.

Theorem (LM):

There exist $\epsilon > 0, d < \infty$ such that

$$\lim_{n\to\infty} Pr\left[Y \in \mathcal{Y}(n, d) : h_1(Y) > \epsilon\right] = 1.$$

Remark: $\epsilon = 10^{-11}$ and $d = 10^{11}$ will do.

Idea of Proof

Fix $0 < c < 1$ and let $\phi \in C^1(T_n; \mathbb{F}_2)$.

$$\phi \text{ is } \begin{cases} c - \text{small} & \text{if } \|[\phi]\| \leq cn^2 \\ c - \text{large} & \text{if } \|[\phi]\| \geq cn^2 \end{cases}$$

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c-Large Cochains

Expansion is obtained by means of a new large deviations bound for the probability space $\mathcal{L}_n$ of Latin squares.

2-Expansion and Spectral Gap

Notation

For a complex $T_n^{(1)} \subset Y \subset T_n$ let:

$Y_v = lk(Y, v) =$ the link of $v \in V$.

$\mu_v =$ spectral gap of the $n \times n$ bipartite graph Y_v.

$\tilde{\mu} = \min_{v \in V} \mu_v$.

$d = D_1(Y) =$ maximum edge degree in Y.

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Theorem (LM):

If $\|[\phi]\| \leq cn^2$ then

$$\|d_1 \phi\| \geq \left(\frac{(1 - c^{1/3})\tilde{\mu}}{2} - \frac{d}{3} \right) \|[\phi]\|.$$

Spectral Gap of Random Graphs

Random Bipartite Graphs

$\tilde{\pi} = (\pi_1, \ldots, \pi_d) \in \mathbb{S}_n^d$ defines a graph $G = G(\tilde{\pi})$ by

$$E(G) = \{\ (i, \pi_j(i))\ :\ 1 \leq i \leq n, 1 \leq j \leq d\ \} \subset [n]^2.$$

$\mathcal{G}(n, d)$= uniform probability space $\{G(\tilde{\pi}) : \tilde{\pi} \in \mathbb{S}_n^d\}$.

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Theorem (Friedman):

For a fixed $d \geq 100$:

$$Pr[G \in \mathcal{G}(n,d)\ :\ \mu_2(G) > d - 3\sqrt{d}] = 1 - O(n^{-2}).$$

Expansion of c-Small Cochains

Links as Random Graphs

Let $Y = Y(\underline{L}^d)$ be a random complex in $\mathcal{Y}(n, d)$.
Then $Y_v = lk(Y, v)$ is a random graph in $\mathcal{G}(n, d)$.
Therefore

$$Pr[\ \tilde{\mu} \geq d - 3\sqrt{d}\ ] = 1 - O(n^{-1}).$$

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Corollary:

Let $d \geq 100$ and $c < 10^{-3}$. If $\|[\phi]\| \leq cn^2$ then

$$\begin{aligned}\frac{\|d_1\phi\|}{\|[\phi]\|} &\geq \frac{(1-c^{1/3})\tilde{\mu}}{2} - \frac{d}{3} \\ &\geq \frac{(1-c^{1/3})(d-3\sqrt{d})}{2} - \frac{d}{3} > 1.\end{aligned}$$

Large Deviations for Latin Squares

The Random Variable $f_{\mathcal{E}}$

$\mathcal{E}$ - a family of 2-simplices of T_n, $|\mathcal{E}| \geq cn^3$.
For a Latin square $L \in \mathcal{L}_n$ let

$$f_{\mathcal{E}}(L) = |Y(L) \cap \mathcal{E}|.$$

Then

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Theorem (LM):

For all $n \geq n_0(c)$

$$Pr[L \in \mathcal{L}_n \ : \ f_{\mathcal{E}}(L) < 10^{-3}c^2n^2] < e^{-10^{-3}c^2n^2}.$$

Remarks on the Proof

For $[a_i, b_j, c_k] \in \mathcal{E}$ define a $0-1$ random variable Z_{ijk} on $\mathcal{L}_n$ by $Z_{ijk}(L) = 1$ iff $\pi_i(j) = k$. Then

$$f_{\mathcal{E}} = \sum_{ijk} Z_{ijk}.$$

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The Z_{ijk} are however far from independent and thus one cannot apply standard Chernoff type bounds.

The actual proof uses a different approach, relying among other things on Brégman's permanent bound and on the classical asymptotic enumeration of Latin squares:

$$|\mathcal{L}_n| = \left(\frac{(1+o(1))n}{e^2}\right)^{n^2}.$$

Expansion of c-Large Cochains I

Expansion in T_n

Theorem (Dotterrer and Kahle): $h_1(T_n) \geq \frac{n}{5}$.
Therefore, if $\phi \in C^1(T_n)$ then

$$\mathcal{E} = \{\sigma \in T_n(2) : d_1\phi(\sigma) \neq 0\}$$

satisfies

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Expansion in $Y(L)$

If $L \in \mathcal{L}_n$ then $\|d_1\phi\|_{Y(L)} = |Y(L) \cap \mathcal{E}| = f_{\mathcal{E}}(L)$.
Hence, by the large deviation bound:

$$Pr[L \in \mathcal{L}_n : \|d_1\phi\|_{Y(L)} < \delta c^2 n^2] < e^{-\delta c^2 n^2}.$$

for some absolute $\delta > 0$.

Expansion of c-Large Cochains II

Let $\underline{L}^d = (L_1, \ldots, L_d) \in \mathcal{L}_n^d$. Then

$$\|d_1\phi\|_{Y(\underline{L}^d)} = |Y(\underline{L}^d) \cap \mathcal{E}| \geq \max_{1 \leq i \leq d} f_{\mathcal{E}}(L_i).$$

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Since $|C^1(T_n; \mathbb{F}_2)| = 2^{3n^2}$ it follows that

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Choosing d large it follows that a.a.s. for all c-large ϕ:

$$\frac{\|d_1\phi\|_{Y(\underline{L}^d)}}{\|\phi\|} \geq \frac{\delta c^2 n^2}{3n^2} = \frac{\delta c^2}{3}.$$

Homological Connectivity on $\mathcal{Y}(n, d)$

Corollary:

There exists $d < \infty$ such that

$$\lim_{n \to \infty} Pr\left[Y \in \mathcal{Y}(n, d) : H_1(Y; \mathbb{F}_2) = 0\right] = 1.$$

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Claim:

$$\lim_{n\to\infty} Pr[H_1(Y(L_1, L_2, L_3); \mathbb{F}_2) \neq 0] \geq 1 - \frac{17e^{-3}}{2} \doteq 0.57.$$

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Claim:

$$\lim_{n\to\infty} Pr[H_1(Y(L_1, L_2, L_3); \mathbb{F}_2) \neq 0] \geq 1 - \frac{17e^{-3}}{2} \doteq 0.57.$$

Theorem (Garland): If in a 2-dimensional complex Y all vertex links have sufficiently large spectral gaps then $H_1(Y; \mathbb{R}) = 0$.

Corollary: If $d \geq 100$ then $H_1(Y; \mathbb{R}) = 0$ a.a.s. for $Y \in \mathcal{Y}(n, d)$.

Topological Overlap Property for $\mathcal{Y}(n, d)$

Corollary:

There exist $\delta > 0$ and d such that $Y \in \mathcal{Y}(n, d)$ a.a.s. satisfies the following:
For any continuous map $f : Y \to \mathbb{R}^2$ there exists $p \in \mathbb{R}^2$ such that

$$\gamma_Y(p) \geq \delta n^2.$$

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- Are Ramanujan complexes high dimensional expanders?
- The model $\mathcal{Y}(n, d)$ generalizes to higher dimensions. Does the Theorem remain true there?
- Find the minimal d for which Theorem 1 holds.