Generalized Flow-Cut Dualities

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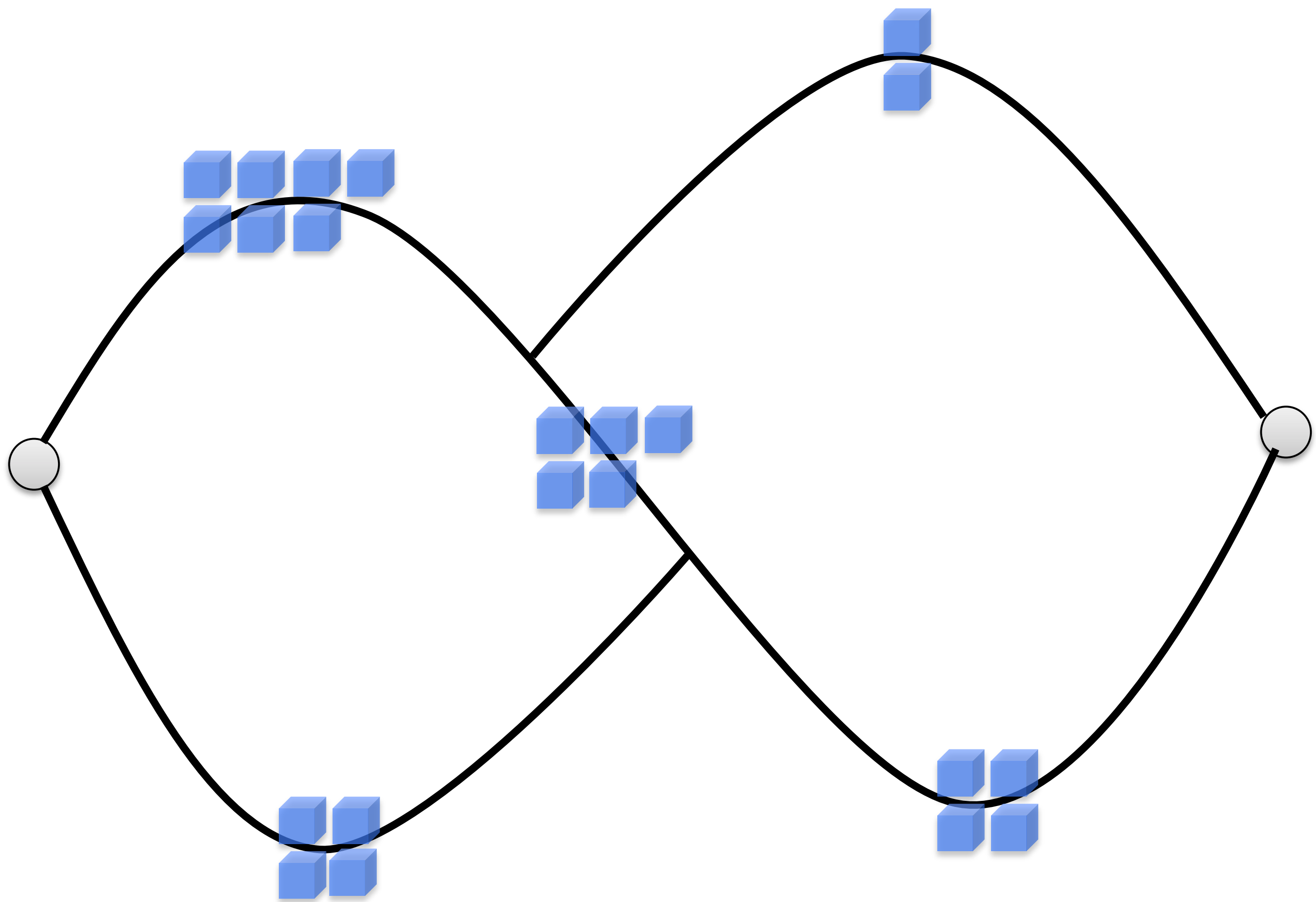
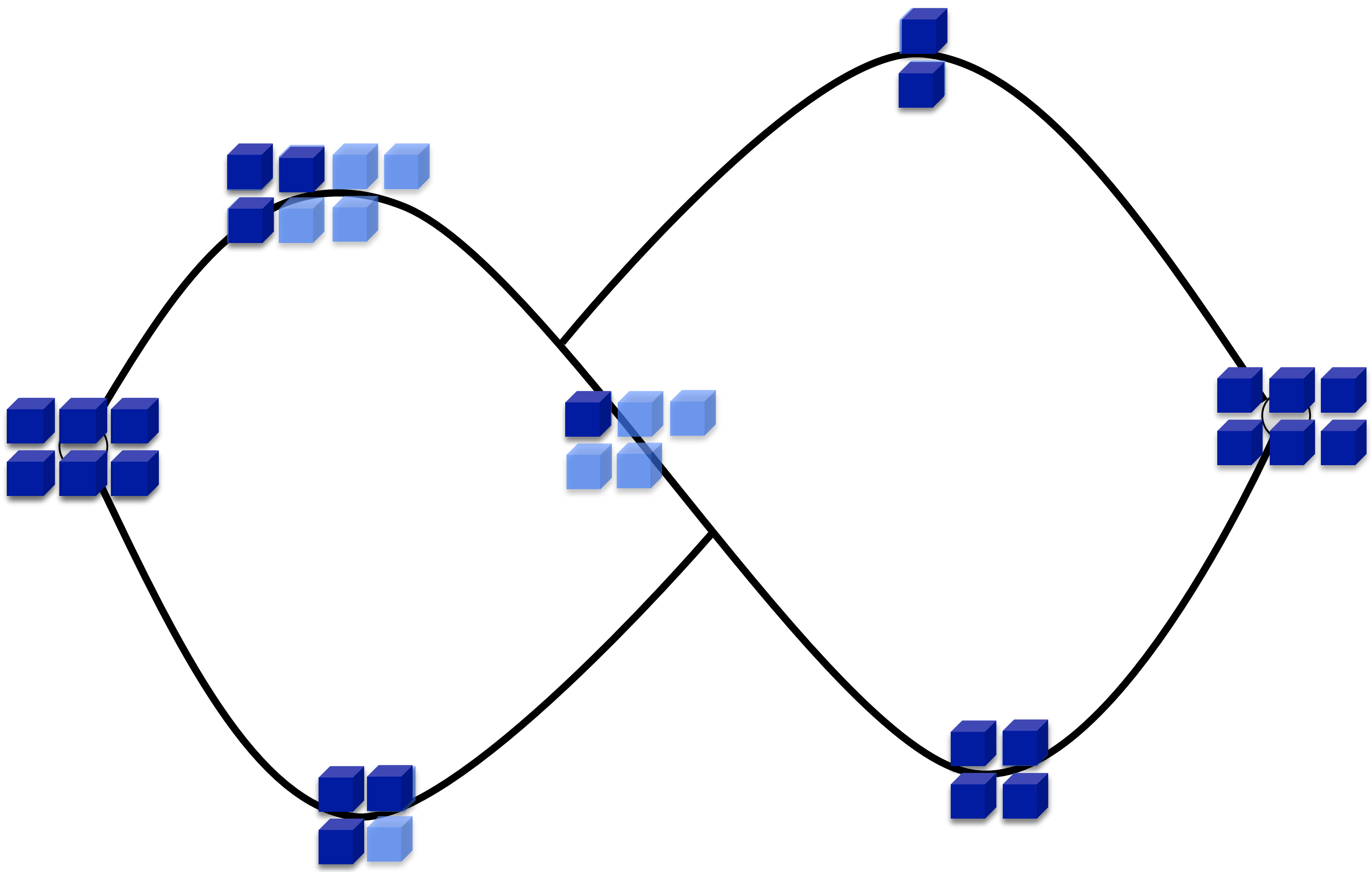
MAX FLOW = MIN CUT

The setting is a weighted digraph G with distinguished start and end points a,b.



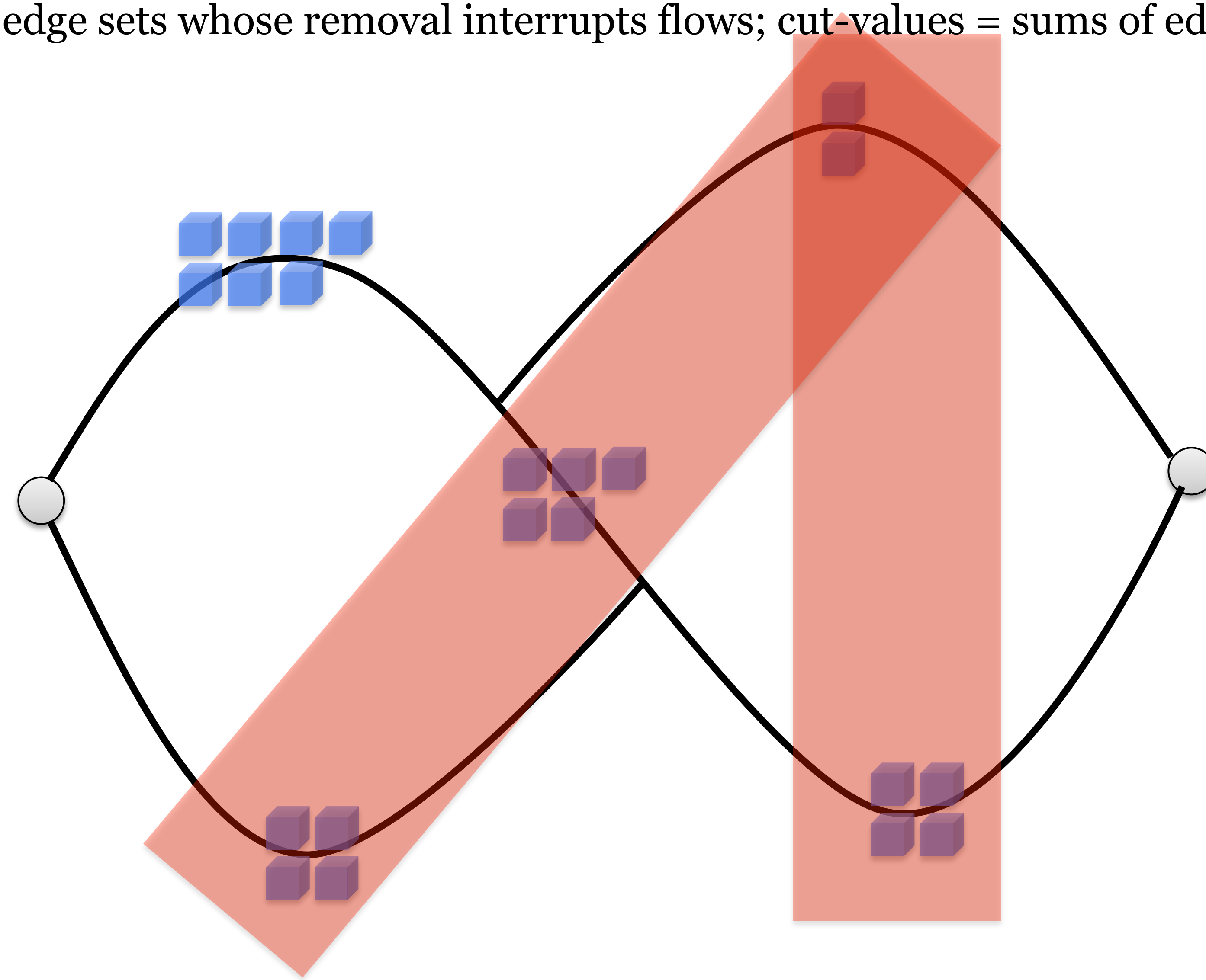
MAX FLOW = MIN CUT

Flows = integral 1-cycles whose coefficients are non-negative and obey constraints.

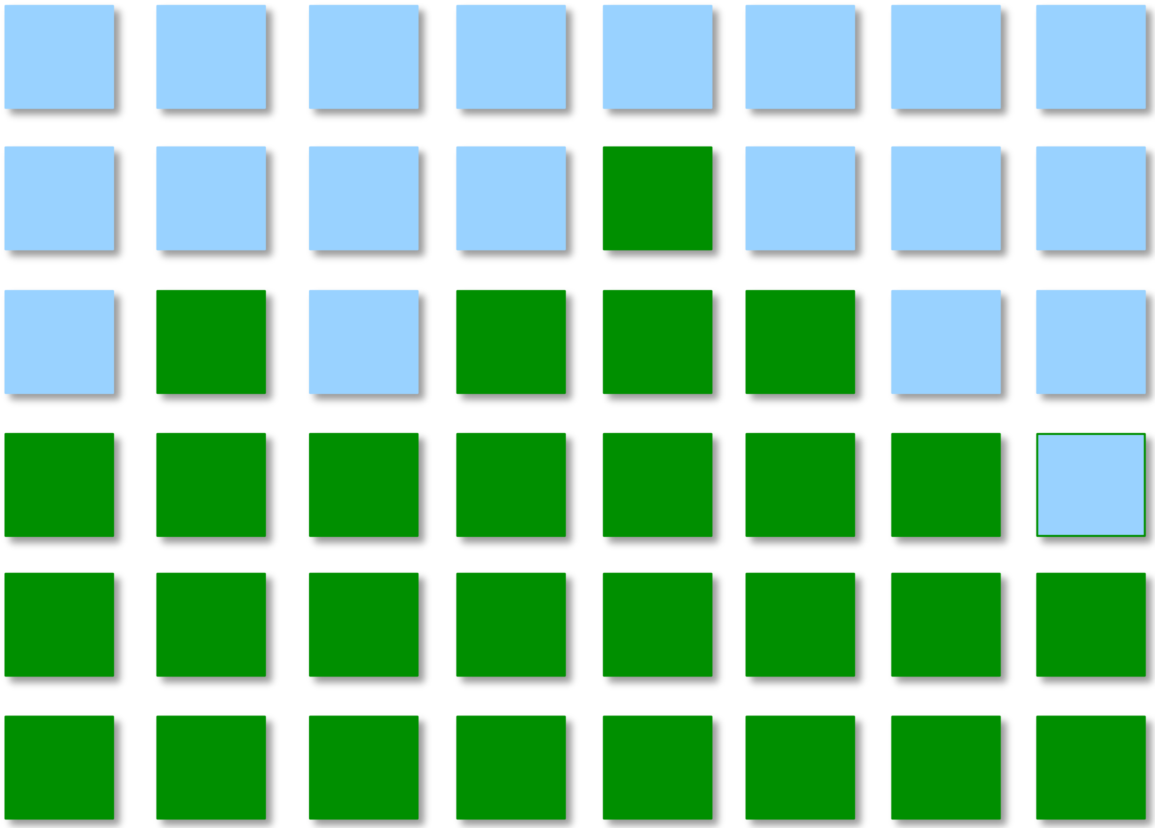


MAX FLOW = MIN CUT

Cuts = edge sets whose removal interrupts flows; cut-values = sums of edge weights.



cut minimization



minimum energy partition of picture into two pieces

ZERO SEMIMODULES

zero semimodule = commutative monoid w/ **algebraic zero**

∞: ∞+x=∞ for all x

[0,5]

$$x \oplus y = \min(x + y, 5)$$

[0,5+δ] = $\mathbf{R}^+$ / (5,∞)

$$[x] + [y] = [x + y]$$

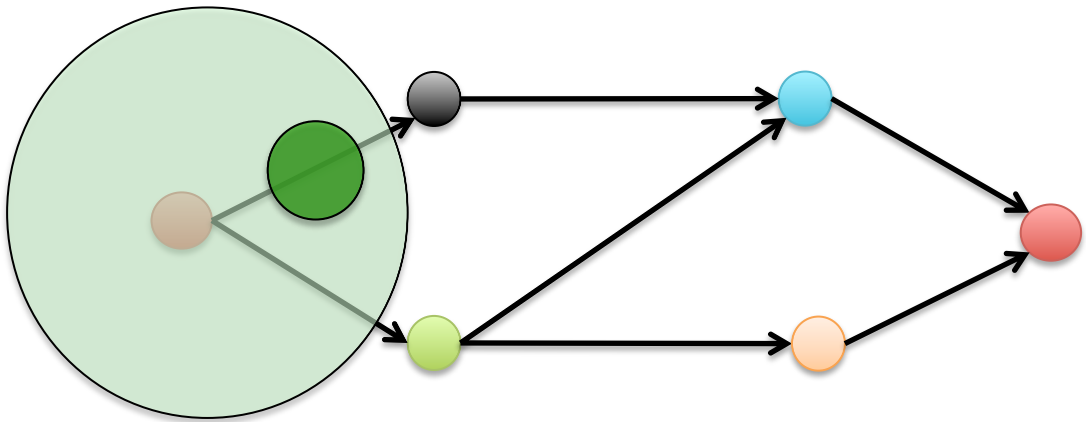
[a,b]

$$a \oplus b = \max(a, b)$$

logical propositions under ∧

T = unit   F = algebraic zero

(CO)SHEAF OF SEMIMODULES



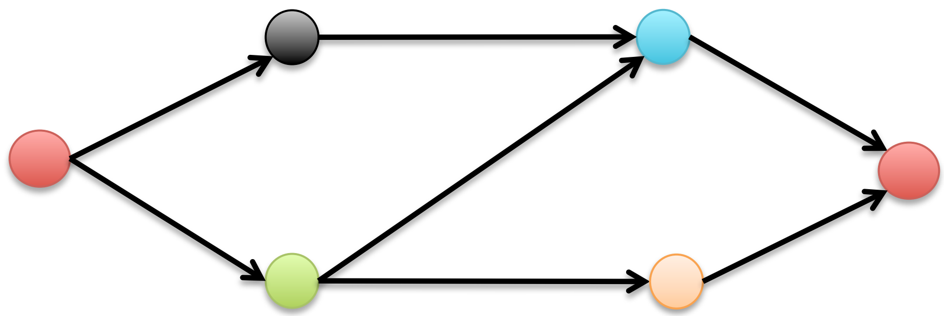
cellular sheaf (on a digraph) = function F assigning zero semimodules to vertices and edges and zero homomorphism from F(v) to F(e) for each inclusion of vertex v into edge e.

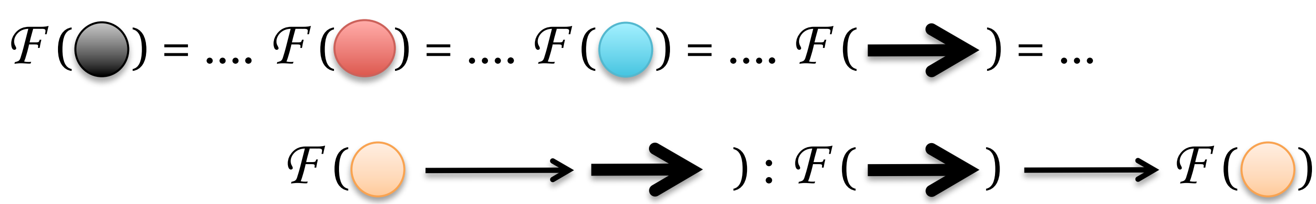
cellular cosheaf (on a digraph) = Function F … from F(e) to F(v) …

survey of cellular (co)sheaves by Justin Curry

GENERALIZED FLOWS

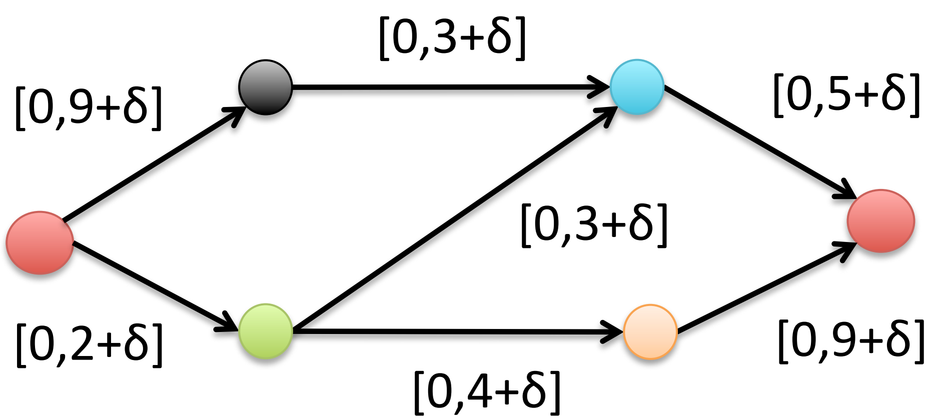
$$\underbrace{H_1(G}_{\text{digraph}};\underbrace{\mathcal{F}) \dashrightarrow \bigoplus_{\sigma\in E_G}\mathcal{F}\langle\sigma\rangle}_{\text{cellular sheaf of zero semimodules}} \;\underset{\gamma\mapsto\gamma_1}{\overset{\gamma\mapsto\gamma_0}{\rightrightarrows}}\; \bigoplus_{v\in V_G}\mathcal{F}(v)$$

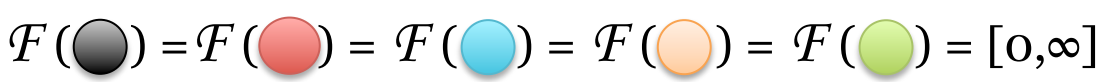




ORDINARY FLOWS

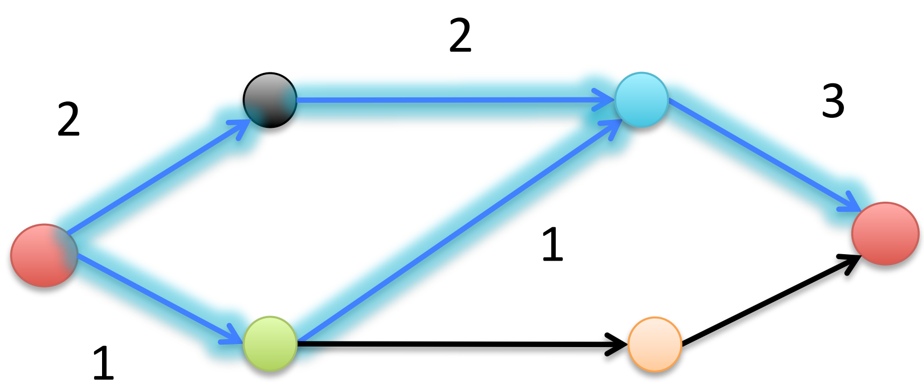
$$H_1(G;\mathcal{F}) \dashrightarrow \bigoplus_{\sigma\in E_G}\mathcal{F}\langle\sigma\rangle \underset{\gamma\mapsto\gamma_1}{\overset{\gamma\mapsto\gamma_0}{\rightrightarrows}} \bigoplus_{v\in V_G}\mathcal{F}(v)$$





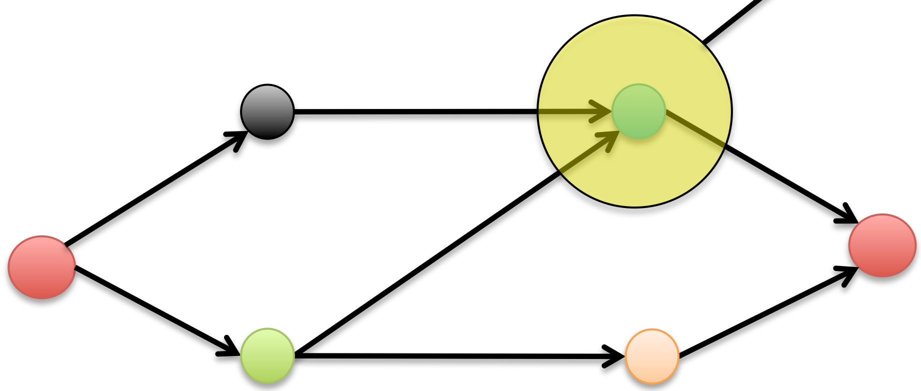
ORDINARY FLOWS

$$H_1(G;\mathcal{F}) \dashrightarrow \bigoplus_{\sigma\in E_G} \mathcal{F}\langle\sigma\rangle \underset{\gamma\mapsto\gamma_1}{\overset{\gamma\mapsto\gamma_0}{\rightrightarrows}} \bigoplus_{v\in V_G} \mathcal{F}(v)$$



CUT CHARACTERIZATION

$$\mathcal{O}(U) = H_n(X, X - U; \mathbb{N}) = \mathbb{N} \oplus \mathbb{N} \oplus \mathbb{N}$$



O: local directed homology (orientation) sheaf

CUT CHARACTERIZATION

$$\begin{tikzcd}
\mathcal{O}(G) \arrow[r] \arrow[dr] & \mathcal{O}(U) \arrow[d, dotted] \\
& \mathcal{O}_{\star}
\end{tikzcd}$$

point at which flow-value is taken

A minimal cut-set U is a minimal subset of edges such that there exists a unique natural map making the diagram commute.

[set of local flows at U] = set of feasible values of such local flows is a quotient determined by the dotted map

WEAK DUALITY

$$[\lim_U H_1((X, X-U);\mathcal{F})] = [H_1(X;\mathcal{F})] \dashrightarrow \lim_U [H_1((X, X-U);\mathcal{F})]$$

trivial case of
Poincare Duality

intersection of cut-capacities

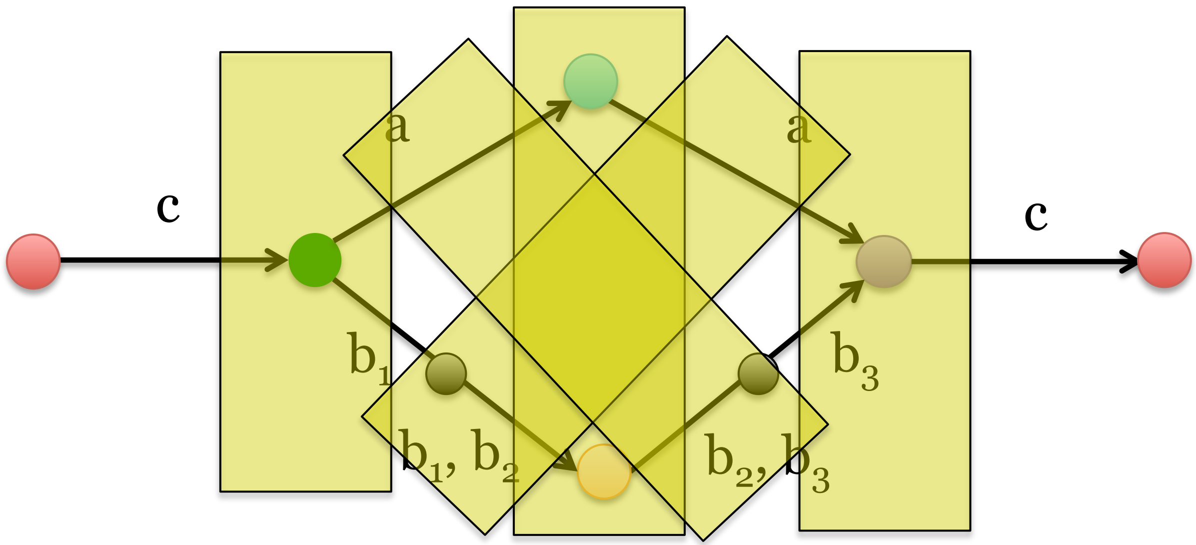
feasible flow-values subset of
feasible cut-capacities

$$H^p(X;\mathcal{F}\otimes\mathcal{O}) \cong H_{n-p}(X;\mathcal{F})$$

$$\begin{tikzcd}
\mathscr{O}(X) \arrow[r] \arrow[dr] & \mathscr{O}(U) \arrow[d, dotted] \\
 & \mathscr{O}_\star,
\end{tikzcd}$$

DUALITY GAP

$$H_1(G;\mathcal{F}) \dashrightarrow \bigoplus_{\sigma\in E_G}\mathcal{F}\langle\sigma\rangle \underset{\gamma\mapsto\gamma_1}{\overset{\gamma\mapsto\gamma_0}{\rightrightarrows}} \bigoplus_{v\in V_G}\mathcal{F}(v)$$



$M=\langle a,b_1,b_2,b_3,c \mid a+b_i=c\rangle$

feasible flow-values = 0 **feasible cut-capacities** = <c>

(illustrated is the cosheaf associated to a sheaf)

POINCARE DUALITY

$$H^p(X;\mathcal{F}\otimes\mathcal{O})\cong H_{n-p}(X;\mathcal{F})$$

weak
homology
manifold

local n-homology sheaf

*

Abelian sheaf (co)homology [Bredon]

* N[sing -] should actually be replaced by its localization at its (n-1)-skeleton.

POINCARE DUALITY

$$H^p(X;\mathcal{F}\otimes\mathcal{O}) \cong H_{n-p}(X;\mathcal{F})$$

generalized spacetimes

local directed n-homology sheaf

$$H_n(X;\mathcal{F}) = \pi_0\Gamma_c\left(\Omega^n\left(\mathbb{N}[singX]\big/\mathbb{N}[sing\,(X\setminus -)]\right)\otimes\mathcal{F}\right)^*$$

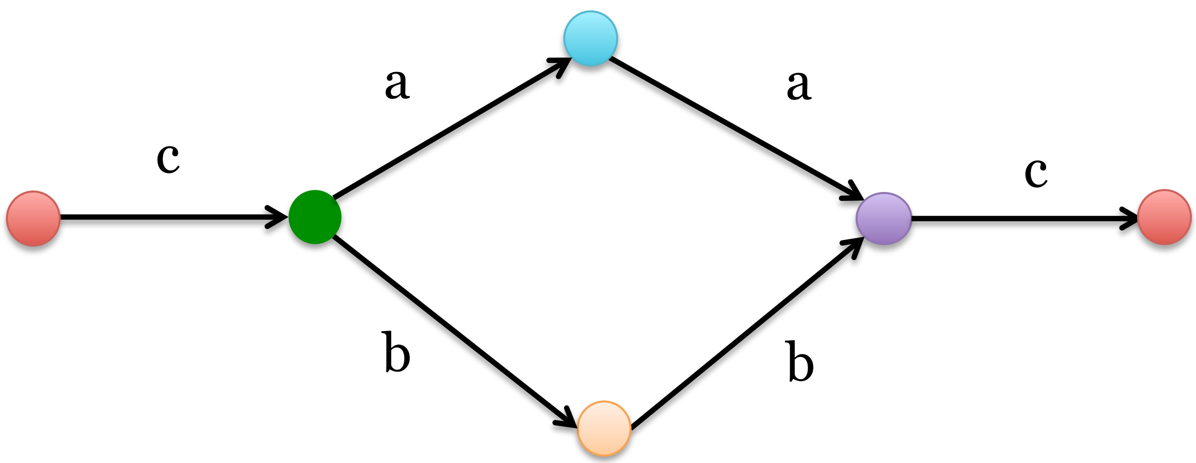
$$H^n(X;\mathcal{F}) = \pi_0\Gamma_c\,(hom(\mathbb{N}[sing\,-],K(\mathbb{N},n))\otimes\mathcal{F})$$

directed cycle sheaf

* N[sing -] should actually be replaced by its localization at its (n-1)-skeleton.

EXAMPLE

$H_1=0$



$M=\langle a,b,c \mid a+b=c\rangle$

failure for revised definition of homology to capture all flows

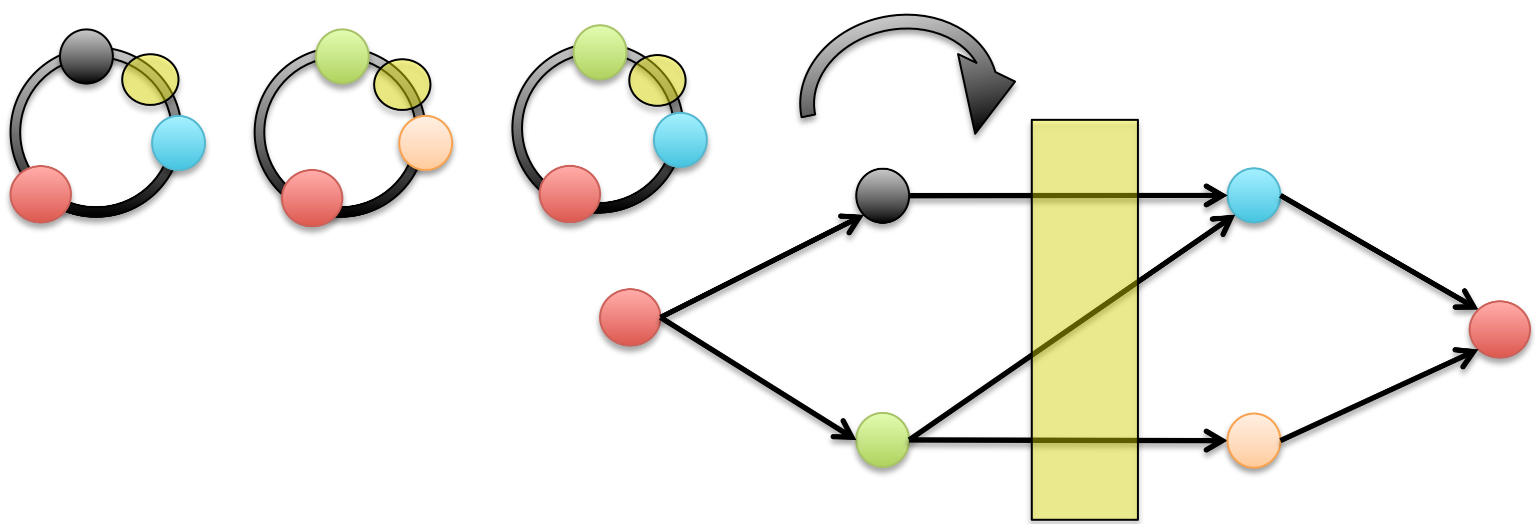
(illustrated is the cosheaf associated to a sheaf)

FLOW-CUT DUALITY

THM: For all sheaves F on digraphs G,

$$\lim_{U}[H^0(U;\mathcal{F}\otimes\mathcal{O})] \cong [H_1(G;\mathcal{F})]$$

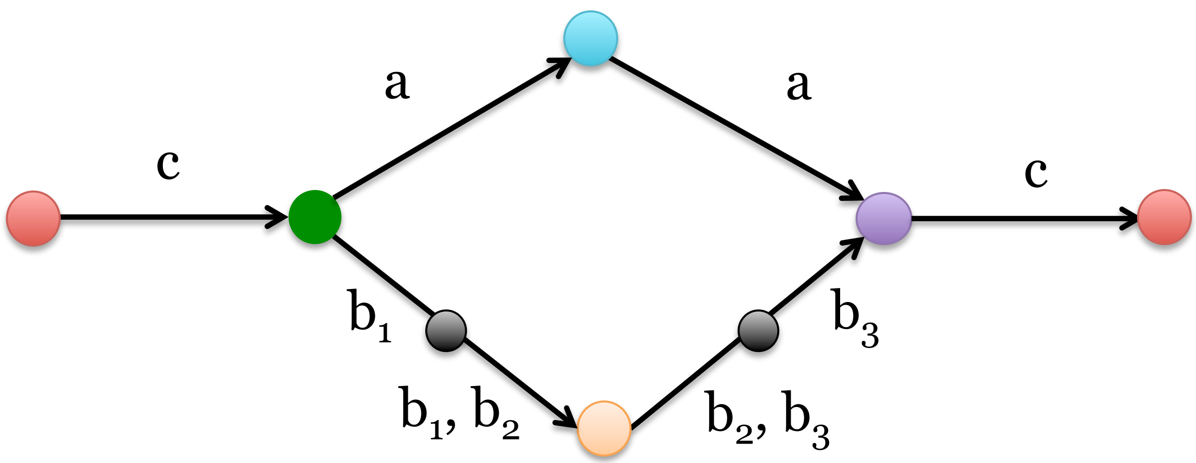
where the limit is taken over cut-sets U*



*abstracting the proof for an algebraic MFMC in Frieze, *Algebraic Flows*

EXAMPLE

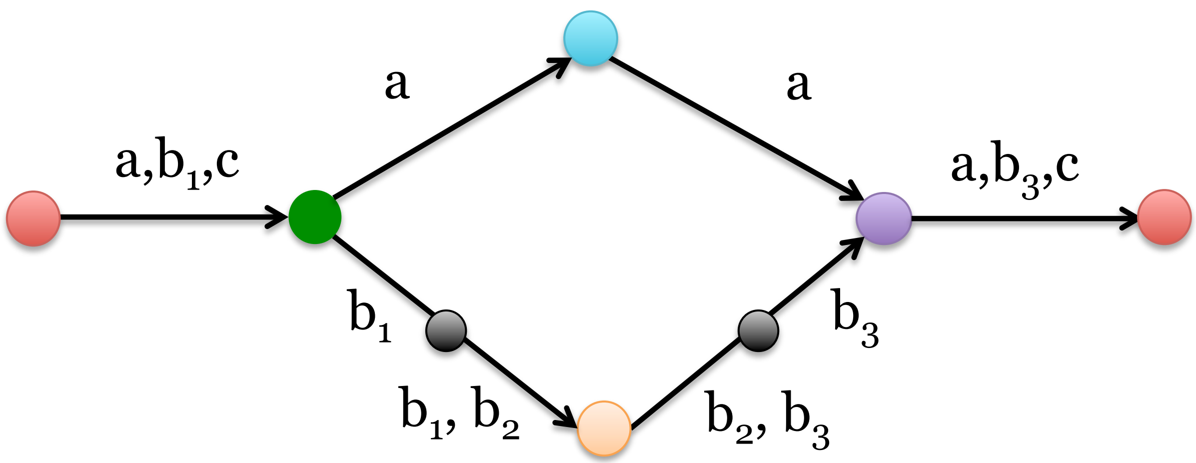
$H_1 = H^0 = 0$ because there are neither local nor global loops



$M = \langle a, b_1, b_2, b_3, c \mid a + b_i = c \rangle$

(illustrated is the cosheaf associated to a sheaf)

EXAMPLE



$M = \langle a, b_1, b_2, b_3, c \mid a + b_i = c \rangle$

feasible flow-values = $\langle a \rangle$ **feasible cut-capacities** = $\langle a \rangle$

(illustrated is the cosheaf associated to a sheaf)

FUTURE DIRECTIONS

smooth setting	de Rham formulations of directed cohomology for smooth manifolds with distinguished vector fields
algorithms for generalized max flows	finding flat and soft resolutions and then adapting classical algorithms (e.g. Ford-Fulkerson)
lp duality as higher mfmc	reformulating general primal problems as higher homology (= higher flows)
network coding, multi-commodities, stochastic capacities	ongoing, joint with Rob Ghrist, Greg Henselmen