An introduction to finite frames

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  - Major open problems

Part I:
Finite Frames

A Brief History of Frame Theory

1950s: **Frames** are introduced to study nonharmonic Fourier series.
Infinite-dimensional generalization of standard linear algebra.

1960s-1970s: "Frames" is an obscure term used by harmonic analysts.
Time-frequency analysis routinely used in real-world applications.

1980s-1990s: **Wavelets** (time-scale analysis) invented to address shortcomings of time-frequency analysis.
Frame theory used to compare these two competing methods.
Frames popularized as "painless nonorthogonal expansions."

2000s-2010s: **Finite frame theory** developed to study packing and covering problems in Euclidean geometry.
It overlaps with **compressed sensing**, which is invented to address shortcomings of wavelets.

Common theme: In what ways (and to what degree) can nonorthonormal vectors behave like orthonormal vectors?

Matrix Notation

Definition: Let M, N be positive integers and let either $\mathbb{F} = \mathbb{R}$ or $\mathbb{F} = \mathbb{C}$. Given N vectors $\{\boldsymbol{\varphi}_n\}_{n=1}^N$ in $\mathbb{F}^M$, consider the

- $M \times N$ **synthesis operator** $\boldsymbol{\Phi} = \begin{bmatrix} \boldsymbol{\varphi}_1 & \cdots & \boldsymbol{\varphi}_N \end{bmatrix}$,

- $N \times M$ **analysis operator** $\boldsymbol{\Phi}^* = \begin{bmatrix} \boldsymbol{\varphi}_1^* \\ \vdots \\ \boldsymbol{\varphi}_N^* \end{bmatrix}$,

- $M \times M$ **frame operator** $\boldsymbol{\Phi}\boldsymbol{\Phi}^* = \boldsymbol{\varphi}_1\boldsymbol{\varphi}_1^* + \cdots + \boldsymbol{\varphi}_N\boldsymbol{\varphi}_N^*$,

- $N \times N$ **Gram matrix** $\boldsymbol{\Phi}^*\boldsymbol{\Phi} = \begin{bmatrix} \boldsymbol{\varphi}_1^*\boldsymbol{\varphi}_1 & \cdots & \boldsymbol{\varphi}_1^*\boldsymbol{\varphi}_N \\ \vdots & \ddots & \vdots \\ \boldsymbol{\varphi}_N^*\boldsymbol{\varphi}_1 & \cdots & \boldsymbol{\varphi}_N^*\boldsymbol{\varphi}_N \end{bmatrix}$.

Orthonormal Bases

Notes:

- Vectors $\{\boldsymbol{\varphi}_n\}_{n=1}^N$ in $\mathbb{F}^M$ are orthonormal if and only if $\boldsymbol{\Phi}^*\boldsymbol{\Phi} = \mathbf{I}$.

- Vectors $\{\boldsymbol{\varphi}_n\}_{n=1}^N$ are an orthonormal basis for $\mathbb{F}^M$ if and only if they're orthonormal and $M = N$.

- In that case, $\boldsymbol{\Phi}$ is square and $\boldsymbol{\Phi}^* = \boldsymbol{\Phi}^{-1}$ implying that $\forall \mathbf{x} \in \mathbb{F}^M$,

$$\mathbf{x} = \boldsymbol{\Phi}\boldsymbol{\Phi}^*\mathbf{x} = \left(\sum_{n=1}^N \boldsymbol{\varphi}_n\boldsymbol{\varphi}_n^*\right)\mathbf{x} = \sum_{n=1}^N (\boldsymbol{\varphi}_n^*\mathbf{x})\boldsymbol{\varphi}_n.$$

- This implies the Pythagorean theorem: $\forall \mathbf{x} \in \mathbb{F}^M$,

$$\|\mathbf{x}\|^2 = \mathbf{x}^*\mathbf{x} = \mathbf{x}^*\boldsymbol{\Phi}\boldsymbol{\Phi}^*\mathbf{x} = \mathbf{x}^*\left(\sum_{n=1}^N \boldsymbol{\varphi}_n\boldsymbol{\varphi}_n^*\right)\mathbf{x} = \sum_{n=1}^N |\boldsymbol{\varphi}_n^*\mathbf{x}|^2.$$

Finite Frames

- Now suppose your real-world application prohibits you from having $\{\varphi_n\}_{n=1}^N$ be an orthonormal basis for $\mathbb{F}^M$.

- As long as $\{\varphi_n\}_{n=1}^N$ spans $\mathbb{F}^M$, you can still "painlessly" expand any $\mathbf{x}$ in terms of them: in this case, $\mathbf{\Phi\Phi}^*$ is invertible and so

$$\mathbf{x} = \mathbf{\Phi\Phi}^*(\mathbf{\Phi\Phi}^*)^{-1}\mathbf{x} = \mathbf{\Phi\Psi}^*\mathbf{x} = \left(\sum_{n=1}^N \varphi_n \psi_n^*\right)\mathbf{x} = \sum_{n=1}^N (\psi_n^*\mathbf{x})\varphi_n.$$

- This expansion is numerically stable when $\mathbf{\Phi}$ is well conditioned, i.e. when $\{\varphi_n\}_{n=1}^N$ satisfies a relaxed Pythagorean theorem:

$$\alpha\|\mathbf{x}\|^2 \leq \mathbf{x}^*\mathbf{\Phi\Phi}^*\mathbf{x} = \sum_{n=1}^N |\varphi_n^*\mathbf{x}|^2 \leq \beta\|\mathbf{x}\|^2, \quad \forall \mathbf{x} \in \mathbb{F}^M,$$

  for "close" scalars $0 < \alpha \leq \beta < \infty$. Here, we call $\{\varphi_n\}_{n=1}^N$ a **frame** for $\mathbb{F}^M$ with **lower** and **upper frame bounds** α and β, respectively.

Tight Frames

- We say $\{\varphi_n\}_{n=1}^N$ is a **tight frame** for $\mathbb{F}^M$ if $\mathbf{\Phi}$ is optimally well conditioned, namely when there exists $\alpha > 0$ such that

$$\alpha\|\mathbf{x}\|^2 = \mathbf{x}^*\mathbf{\Phi}^*\mathbf{\Phi}\mathbf{x} = \sum_{n=1}^{N} |\varphi_n^*\mathbf{x}|^2, \quad \forall \mathbf{x} \in \mathbb{F}^M.$$

- This is equivalent to $\mathbf{\Phi}\mathbf{\Phi}^* = \alpha\mathbf{I}$, i.e. to when the *rows* of $\mathbf{\Phi}$ are orthogonal and have constant norm.

- **Naimark's Theorem:** Every tight frame is a scalar multiple of an orthogonal projection of an orthonormal basis.

Example: 6×16 Tight Frame

$$\begin{bmatrix}
1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&1&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&1&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&1&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&1&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&1&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&1&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&1&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&1&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&1&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&1&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&1&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&1&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&1&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&1
\end{bmatrix}$$

Example: 6×16 Tight Frame

$$\begin{bmatrix}
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1
\end{bmatrix}$$

Example: 6×16 Tight Frame

$$\boldsymbol{\Phi} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Example: 6×16 Tight Frame

$$\mathbf{\Phi} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Question: This is one of many tight frames of 16 vectors in $\mathbb{R}^6$... can we find others that are even more like orthonormal bases in some sense?

Example: Better 6×16 Tight Frame

$$\begin{bmatrix}
+ & + & + & + & + & + & + & + & + & + & + & + & + & + & + & + \\
+ & - & + & - & + & - & + & - & + & - & + & - & + & - & + & - \\
+ & + & - & - & + & + & - & - & + & + & - & - & + & + & - & - \\
+ & - & - & + & + & - & - & + & + & - & - & + & + & - & - & + \\
+ & + & + & + & - & - & - & - & + & + & + & + & - & - & - & - \\
+ & - & + & - & - & + & - & + & + & - & + & - & - & + & - & + \\
+ & + & - & - & - & - & + & + & + & + & - & - & - & - & + & + \\
+ & - & - & + & - & + & + & - & + & - & - & + & - & + & + & - \\
+ & + & + & + & + & + & + & + & - & - & - & - & - & - & - & - \\
+ & - & + & - & + & - & + & - & - & + & - & + & - & + & - & + \\
+ & + & - & - & + & + & - & - & - & - & + & + & - & - & + & + \\
+ & - & - & + & + & - & - & + & - & + & + & - & - & + & + & - \\
+ & + & + & + & - & - & - & - & - & - & - & - & + & + & + & + \\
+ & - & + & - & - & + & - & + & - & + & - & + & + & - & + & - \\
+ & + & - & - & - & - & + & + & - & - & + & + & + & + & - & - \\
+ & - & - & + & - & + & + & - & - & + & + & - & + & - & - & +
\end{bmatrix}$$

Notation: "$+$" $= 1$, "$-$" $= -1$

Example: Better 6×16 Tight Frame

$$
\begin{bmatrix}
+ & + & + & + & + & + & + & + & + & + & + & + & + & + & + & + \\
+ & - & + & - & + & - & + & - & + & - & + & - & + & - & + & - \\
+ & + & - & - & + & + & - & - & + & + & - & - & + & + & - & - \\
+ & - & - & + & + & - & - & + & + & - & - & + & + & - & - & + \\
+ & + & + & + & - & - & - & - & + & + & + & + & - & - & - & - \\
+ & - & + & - & - & + & - & + & + & - & + & - & - & + & - & + \\
+ & + & - & - & - & - & + & + & + & + & - & - & - & - & + & + \\
+ & - & - & + & - & + & + & - & + & - & - & + & - & + & + & - \\
+ & + & + & + & + & + & + & + & - & - & - & - & - & - & - & - \\
+ & - & + & - & + & - & + & - & - & + & - & + & - & + & - & + \\
+ & + & - & - & + & + & - & - & - & - & + & + & - & - & + & + \\
+ & - & - & + & + & - & - & + & - & + & + & - & - & + & + & - \\
+ & + & + & + & - & - & - & - & - & - & - & - & + & + & + & + \\
+ & - & + & - & - & + & - & + & - & + & - & + & + & - & + & - \\
+ & + & - & - & - & - & + & + & - & - & + & + & + & + & - & - \\
+ & - & - & + & - & + & + & - & - & + & + & - & + & - & - & +
\end{bmatrix}
$$

Example: Better 6×16 Tight Frame

$$\boldsymbol{\Phi} = \frac{1}{\sqrt{6}} \begin{bmatrix} + & + & + & + & + & + & + & + & + & + & + & + & + & + & + & + \\ + & - & + & - & + & - & + & - & + & - & + & - & + & - & + & - \\ + & + & - & - & + & + & - & - & + & + & - & - & + & + & - & - \\ + & - & - & + & + & - & - & + & + & - & - & + & + & - & - & + \\ + & + & + & + & - & - & - & - & + & + & + & + & - & - & - & - \\ + & - & + & - & - & + & - & + & + & - & + & - & - & + & - & + \end{bmatrix}$$

Example: Better 6×16 Tight Frame

$$
\mathbf{\Phi} = \frac{1}{\sqrt{6}} \begin{bmatrix}
+ & + & + & + & + & + & + & + & + & + & + & + & + & + & + & + \\
+ & - & + & - & + & - & + & - & + & - & + & - & + & - & + & - \\
+ & + & - & - & + & + & - & - & + & + & - & - & + & + & - & - \\
+ & - & - & + & + & - & - & + & + & - & - & + & + & - & - & + \\
+ & + & + & + & - & - & - & - & + & + & + & + & - & - & - & - \\
+ & - & + & - & - & + & - & + & + & - & + & - & - & + & - & +
\end{bmatrix}
$$

Note: All columns are unit norm.

Example: Better 6×16 Tight Frame

$$
\mathbf{\Phi}^*\mathbf{\Phi} = \frac{1}{3}\begin{bmatrix}
3 & 0 & + & 0 & + & 0 & - & 0 & 3 & 0 & + & 0 & + & 0 & - & 0 \\
0 & 3 & 0 & + & 0 & + & 0 & - & 0 & 3 & 0 & + & 0 & + & 0 & - \\
+ & 0 & 3 & 0 & - & 0 & + & 0 & + & 0 & 3 & 0 & - & 0 & + & 0 \\
0 & + & 0 & 3 & 0 & - & 0 & + & 0 & + & 0 & 3 & 0 & - & 0 & + \\
+ & 0 & - & 0 & 3 & 0 & + & 0 & + & 0 & - & 0 & 3 & 0 & + & 0 \\
0 & + & 0 & - & 0 & 3 & 0 & + & 0 & + & 0 & - & 0 & 3 & 0 & + \\
- & 0 & + & 0 & + & 0 & 3 & 0 & - & 0 & + & 0 & + & 0 & 3 & 0 \\
0 & - & 0 & + & 0 & + & 0 & 3 & 0 & - & 0 & + & 0 & + & 0 & 3 \\
3 & 0 & + & 0 & + & 0 & - & 0 & 3 & 0 & + & 0 & + & 0 & - & 0 \\
0 & 3 & 0 & + & 0 & + & 0 & - & 0 & 3 & 0 & + & 0 & + & 0 & - \\
+ & 0 & 3 & 0 & - & 0 & + & 0 & + & 0 & 3 & 0 & - & 0 & + & 0 \\
0 & + & 0 & 3 & 0 & - & 0 & + & 0 & + & 0 & 3 & 0 & - & 0 & + \\
+ & 0 & - & 0 & 3 & 0 & + & 0 & + & 0 & - & 0 & 3 & 0 & + & 0 \\
0 & + & 0 & - & 0 & 3 & 0 & + & 0 & + & 0 & - & 0 & 3 & 0 & + \\
- & 0 & + & 0 & + & 0 & 3 & 0 & - & 0 & + & 0 & + & 0 & 3 & 0 \\
0 & - & 0 & + & 0 & + & 0 & 3 & 0 & - & 0 & + & 0 & + & 0 & 3
\end{bmatrix}
$$

Example: Even Better 6×16 Tight Frame

$$
\begin{bmatrix}
+ & + & + & + & + & + & + & + & + & + & + & + & + & + & + & + \\
+ & - & + & - & + & - & + & - & + & - & + & - & + & - & + & - \\
+ & + & - & - & + & + & - & - & + & + & - & - & + & + & - & - \\
+ & - & - & + & + & - & - & + & + & - & - & + & + & - & - & + \\
+ & + & + & + & - & - & - & - & + & + & + & + & - & - & - & - \\
+ & - & + & - & - & + & - & + & + & - & + & - & - & + & - & + \\
+ & + & - & - & - & - & + & + & + & + & - & - & - & - & + & + \\
+ & - & - & + & - & + & + & - & + & - & - & + & - & + & + & - \\
+ & + & + & + & + & + & + & + & - & - & - & - & - & - & - & - \\
+ & - & + & - & + & - & + & - & - & + & - & + & - & + & - & + \\
+ & + & - & - & + & + & - & - & - & - & + & + & - & - & + & + \\
+ & - & - & + & + & - & - & + & - & + & + & - & - & + & + & - \\
+ & + & + & + & - & - & - & - & - & - & - & - & + & + & + & + \\
+ & - & + & - & - & + & - & + & - & + & - & + & + & - & + & - \\
+ & + & - & - & - & - & + & + & - & - & + & + & + & + & - & - \\
+ & - & - & + & - & + & + & - & - & + & + & - & + & - & - & +
\end{bmatrix}
$$

Example: Even Better 6×16 Tight Frame

$$
\begin{bmatrix}
+ & + & + & + & + & + & + & + & + & + & + & + & + & + & + & + \\
+ & - & + & - & + & - & + & - & + & - & + & - & + & - & + & - \\
+ & + & - & - & + & + & - & - & + & + & - & - & + & + & - & - \\
+ & - & - & + & + & - & - & + & + & - & - & + & + & - & - & + \\
+ & + & + & + & - & - & - & - & + & + & + & + & - & - & - & - \\
+ & - & + & - & - & + & - & + & + & - & + & - & - & + & - & + \\
+ & + & - & - & - & - & + & + & + & + & - & - & - & - & + & + \\
+ & - & - & + & - & + & + & - & + & - & - & + & - & + & + & - \\
+ & + & + & + & + & + & + & + & - & - & - & - & - & - & - & - \\
+ & - & + & - & + & - & + & - & - & + & - & + & - & + & - & + \\
+ & + & - & - & + & + & - & - & - & - & + & + & - & - & + & + \\
+ & - & - & + & + & - & - & + & - & + & + & - & - & + & + & - \\
+ & + & + & + & - & - & - & - & - & - & - & - & + & + & + & + \\
+ & - & + & - & - & + & - & + & - & + & - & + & + & - & + & - \\
+ & + & - & - & - & - & + & + & - & - & + & + & + & + & - & - \\
+ & - & - & + & - & + & + & - & - & + & + & - & + & - & - & +
\end{bmatrix}
$$

Example: Even Better 6×16 Tight Frame

$$\mathbf{\Phi} = \frac{1}{\sqrt{6}} \begin{bmatrix} + & + & + & + & + & + & + & + & + & + & + & + & + & + & + & + \\ + & - & + & - & + & - & + & - & + & - & + & - & + & - & + & - \\ + & - & - & + & + & - & - & + & + & - & - & + & + & - & - & + \\ + & + & + & + & - & - & - & - & + & + & + & + & - & - & - & - \\ + & - & + & - & + & - & + & - & - & + & - & + & - & + & - & + \\ + & - & - & + & - & + & + & - & - & + & + & - & + & - & - & + \end{bmatrix}$$

Example: Even Better 6×16 Tight Frame

$$
\mathbf{\Phi}^*\mathbf{\Phi} = \frac{1}{3}\begin{bmatrix}
3 & - & + & + & + & - & + & - & + & + & + & + & + & - & - & + \\
- & 3 & + & + & - & + & - & + & + & + & + & + & - & + & + & - \\
+ & + & 3 & - & + & - & + & - & + & + & + & + & - & + & + & - \\
+ & + & - & 3 & - & + & - & + & + & + & + & + & + & - & - & + \\
+ & - & + & - & 3 & - & + & + & + & - & - & + & + & + & + & + \\
- & + & - & + & - & 3 & + & + & - & + & + & - & + & + & + & + \\
+ & - & + & - & + & + & 3 & - & - & + & + & - & + & + & + & + \\
- & + & - & + & + & + & - & 3 & + & - & - & + & + & + & + & + \\
+ & + & + & + & + & - & - & + & 3 & - & + & + & + & - & + & - \\
+ & + & + & + & - & + & + & - & - & 3 & + & + & - & + & - & + \\
+ & + & + & + & - & + & + & - & + & + & 3 & - & + & - & + & - \\
+ & + & + & + & + & - & - & + & + & + & - & 3 & - & + & - & + \\
+ & - & - & + & + & + & + & + & + & - & + & - & 3 & - & + & + \\
- & + & + & - & + & + & + & + & - & + & - & + & - & 3 & + & + \\
- & + & + & - & + & + & + & + & + & - & + & - & + & + & 3 & - \\
+ & - & - & + & + & + & + & + & - & + & - & + & + & + & - & 3
\end{bmatrix}
$$

Part II:
Unit Norm Tight Frames

Unit Norm Tight Frames (UNTFs)

Definition: Vectors $\{\boldsymbol{\varphi}_n\}_{n=1}^N$ are a **unit norm tight frame (UNTF)** for $\mathbb{F}^M$ if $\|\boldsymbol{\varphi}_n\| = 1$ for all n and there exists $\alpha > 0$ such that

$$\alpha\mathbf{I} = \boldsymbol{\Phi}\boldsymbol{\Phi}^* = \sum_{n=1}^{N} \boldsymbol{\varphi}_n \boldsymbol{\varphi}_n^*,$$

i.e. if the orthogonal projection operators onto these lines sum to a scalar multiple of the identity.

Note: Here α is necessarily the **redundancy** $\frac{N}{M}$ since

$$M\alpha = \mathrm{Tr}(\alpha\mathbf{I}) = \mathrm{Tr}(\boldsymbol{\Phi}\boldsymbol{\Phi}^*) = \mathrm{Tr}(\boldsymbol{\Phi}^*\boldsymbol{\Phi}) = \sum_{n=1}^{N} \boldsymbol{\varphi}_n^* \boldsymbol{\varphi}_n = N.$$

Questions: For what M and N do UNTFs exist? How many of them are there? What does the set of all $M \times N$ UNTFs look like?

Frame Potential

Theorem: For any unit vectors $\{\varphi_n\}_{n=1}^N$ in $\mathbb{F}^M$,

$$\frac{N(N-M)}{M} \leq \sum_{n=1}^{N} \sum_{\substack{n'=1 \\ n'\neq n}}^{N} |\varphi_n^* \varphi_{n'}|^2 = \|\mathbf{\Phi}^*\mathbf{\Phi} - \mathbf{I}\|_{\mathrm{Fro}}^2$$

with equality if and only if $\{\varphi_n\}_{n=1}^N$ is a UNTF for $\mathbb{F}^M$.

Proof:

$$\begin{aligned} 0 \leq \mathrm{Tr}(\mathbf{\Phi}\mathbf{\Phi}^* - \tfrac{N}{M}\mathbf{I})^2 &= \mathrm{Tr}[(\mathbf{\Phi}\mathbf{\Phi}^*)^2] - 2\tfrac{N}{M}\,\mathrm{Tr}(\mathbf{\Phi}\mathbf{\Phi}^*) + \tfrac{N^2}{M^2}\,\mathrm{Tr}(\mathbf{I}) \\ &= \mathrm{Tr}[(\mathbf{\Phi}^*\mathbf{\Phi})^2] - 2\tfrac{N}{M}\,\mathrm{Tr}(\mathbf{\Phi}^*\mathbf{\Phi}) + \tfrac{N^2}{M} \\ &= \sum_{n=1}^{N}\sum_{n'=1}^{N} |\varphi_n^*\varphi_{n'}|^2 - \tfrac{N^2}{M}. \end{aligned}$$

Theorem: [Benedetto & F 03] *Local* minimizers of this potential are UNTFs, and so they exist for any $N \geq M$.

Example: UNTFs of 5 vectors in $\mathbb{R}^3$

A technique called **spectral tetris** gives the following 3×5 real UNTF:

$$\mathbf{\Phi} = \begin{bmatrix} 1 & \sqrt{\frac{1}{3}} & \sqrt{\frac{1}{3}} & 0 & 0 \\ 0 & \sqrt{\frac{2}{3}} & -\sqrt{\frac{2}{3}} & \sqrt{\frac{1}{6}} & \sqrt{\frac{1}{6}} \\ 0 & 0 & 0 & \sqrt{\frac{5}{6}} & -\sqrt{\frac{5}{6}} \end{bmatrix}.$$

But it's not the only one. For example, we can rotate (multiply $\mathbf{\Phi}$ by a 3×3 orthogonal matrix) to obtain others. But that's not all...

Example: UNTFs of 5 vectors in $\mathbb{R}^3$

Every 3×5 real UNTF consists of 15 real unknowns:

$$\mathbf{\Phi} = \begin{bmatrix} \mathbf{\Phi}(1,1) & \mathbf{\Phi}(1,2) & \mathbf{\Phi}(1,3) & \mathbf{\Phi}(1,4) & \mathbf{\Phi}(1,5) \\ \mathbf{\Phi}(2,1) & \mathbf{\Phi}(2,2) & \mathbf{\Phi}(2,3) & \mathbf{\Phi}(2,4) & \mathbf{\Phi}(2,5) \\ \mathbf{\Phi}(3,1) & \mathbf{\Phi}(3,2) & \mathbf{\Phi}(3,3) & \mathbf{\Phi}(3,4) & \mathbf{\Phi}(3,5) \end{bmatrix}$$

which satisfy a system of 10 quadratic equations:

- 3 row orthogonality conditions,
- 3 row norm conditions,
- 5 column norm conditions (but one of these is redundant).

Modulo the 3-dimensional orthogonal group $\mathrm{O}(3)$, we thus expect a $15 - 10 - 3 = 2$-dimensional set of UNTFs modulo rotations.

Theorem: [Dykema & Strawn 06] If $N > M$ are relatively prime, the set of all $M \times N$ UNTFs modulo $\mathrm{O}(M)$ is a manifold of dimension

$$(N - M - 1)(M - 1).$$

Paulsen Problem

- **Open Problem:** If $\{\varphi_n\}_{n=1}^N$ is "close" to unit norm, and "close" to tight, how close is $\{\varphi_n\}_{n=1}^N$ to a UNTF?

- **Note:** [Bodmann & Casazza 2010] and [Casazza, F & Mixon 2012] give solutions to this problem when M and N are relatively prime. As noted in [Dykema & Strawn 06], this prevents the frame from being **orthodecomposable** (where the variety "crosses itself").

- The fact that the Paulsen problem is open tells us we still do not really know good ways of "moving around" frames in ways that simultaneously control the norms of our vectors and the spectrum of our frame operator.

Eigensteps Motivation (3×5 UNTFs Example Continued)

- Given any unit vectors $\{\varphi_n\}_{n=1}^5$ in $\mathbb{R}^3$, consider their **partial frame operators** (partial sums of their rank-one orthogonal projections):

$$\begin{aligned}
\mathbf{\Phi}_1\mathbf{\Phi}_1^* &= \varphi_1\varphi_1^*,\\
\mathbf{\Phi}_2\mathbf{\Phi}_2^* &= \varphi_1\varphi_1^* + \varphi_2\varphi_2^*,\\
&\ \vdots\\
\mathbf{\Phi}_5\mathbf{\Phi}_5^* &= \varphi_1\varphi_1^* + \varphi_2\varphi_2^* + \varphi_3\varphi_3^* + \varphi_4\varphi_4^* + \varphi_5\varphi_5^*.
\end{aligned}$$

- For every n, consider the **Rayleigh quotient** over the unit sphere:

$$\mathbf{x} \mapsto \mathbf{x}\varphi_n\varphi_n^*\mathbf{x} = |\varphi_n^*\mathbf{x}|^2,$$

  which has a max of 1 at $\pm\varphi_n$ and min of 0 at the "equator."
  We want 5 of these distributions that sum to $\frac{5}{3}$ everywhere.

- The "hot spots" of $\mathbf{x}\mathbf{\Phi}_n\mathbf{\Phi}_n^*\mathbf{x} = \sum_{i=1}^n |\varphi_i^*\mathbf{x}|^2$ are given in terms of the eigenvalues/vectors of $\mathbf{\Phi}_n\mathbf{\Phi}_n^*$...

Eigensteps

Definition: The **eigensteps** of a UNTF $\{\boldsymbol{\varphi}_n\}_{n=1}^N$ for $\mathbb{F}^M$ is the array $\{\lambda_{m,n}\}_{m=1,\ n=0}^{M\quad N}$ where for any n, $\{\lambda_{m,n}\}_{m=1}^M$ is the nondecreasing spectrum of $\boldsymbol{\Phi}_n\boldsymbol{\Phi}_n^* = \sum_{i=1}^n \boldsymbol{\varphi}_i\boldsymbol{\varphi}_i^*$.

Theorem: [Cahill, F, Mixon, Poteet & Strawn 13]
The eigensteps of any UNTF $\{\boldsymbol{\varphi}_n\}_{n=1}^N$ for $\mathbb{F}^M$ satisfy

- $\lambda_{m,0} = 0$ and $\lambda_{m,N} = \frac{N}{M}$ for all $m = 1, \dots, M$;
- $\sum_{m=1}^M \lambda_{m,n} = n$ for all $n = 0, \dots, N$ (trace condition);
- $\lambda_{m+1,n} \leq \lambda_{m,n-1} \leq \lambda_{m,n}$ for all $m = 1, \dots, M$, $n = 1, \dots, N-1$.

Conversely, for any $\{\lambda_{m,n}\}_{m=1,\ n=0}^{M\quad N}$ that satisfies these properties, there exists a UNTF $\{\boldsymbol{\varphi}_n\}_{n=1}^N$ for $\mathbb{F}^M$ with the property that $\{\lambda_{m,n}\}_{m=1}^M$ is the spectrum of $\sum_{i=1}^n \boldsymbol{\varphi}_i\boldsymbol{\varphi}_i^*$ for all $n = 0, \dots, N$. This construction is explicit, and almost unique up to rotations.

Example: Eigensteps of 3×5 UNTFs

$$\begin{array}{cccccc} \lambda_{3,0} & \lambda_{3,1} & \lambda_{3,2} & \lambda_{3,3} & \lambda_{3,4} & \lambda_{3,5} \\ \lambda_{2,0} & \lambda_{2,1} & \lambda_{2,2} & \lambda_{2,3} & \lambda_{2,4} & \lambda_{2,5} \\ \lambda_{1,0} & \lambda_{1,1} & \lambda_{1,2} & \lambda_{1,3} & \lambda_{1,4} & \lambda_{1,5} \end{array}$$

Example: Eigensteps of 3×5 UNTFs

0	$\lambda_{3,1}$	$\lambda_{3,2}$	$\lambda_{3,3}$	$\lambda_{3,4}$	$\frac{5}{3}$
0	$\lambda_{2,1}$	$\lambda_{2,2}$	$\lambda_{2,3}$	$\lambda_{2,4}$	$\frac{5}{3}$
0	$\lambda_{1,1}$	$\lambda_{1,2}$	$\lambda_{1,3}$	$\lambda_{1,4}$	$\frac{5}{3}$

Example: Eigensteps of 3×5 UNTFs

0	$\lambda_{3,1}$	$\lambda_{3,2}$	$\lambda_{3,3}$	$\lambda_{3,4}$	$\frac{5}{3}$
0	$\lambda_{2,1}$	$\lambda_{2,2}$	$\lambda_{2,3}$	$\lambda_{2,4}$	$\frac{5}{3}$
0	$\lambda_{1,1}$	$\lambda_{1,2}$	$\lambda_{1,3}$	$\lambda_{1,4}$	$\frac{5}{3}$
0	1	2	3	4	5

Example: Eigensteps of 3×5 UNTFs

$$
\begin{array}{ccccccccccc}
0 & \leq & \lambda_{3,1} & \leq & \lambda_{3,2} & \leq & \lambda_{3,3} & \leq & \lambda_{3,4} & \leq & \frac{5}{3} \\
 & \nearrow\!\!\!\!/ & & \nearrow\!\!\!\!/ & & \nearrow\!\!\!\!/ & & \nearrow\!\!\!\!/ & & \nearrow\!\!\!\!/ & \\
0 & \leq & \lambda_{2,1} & \leq & \lambda_{2,2} & \leq & \lambda_{2,3} & \leq & \lambda_{2,4} & \leq & \frac{5}{3} \\
 & \nearrow\!\!\!\!/ & & \nearrow\!\!\!\!/ & & \nearrow\!\!\!\!/ & & \nearrow\!\!\!\!/ & & \nearrow\!\!\!\!/ & \\
0 & \leq & \lambda_{1,1} & \leq & \lambda_{1,2} & \leq & \lambda_{1,3} & \leq & \lambda_{1,4} & \leq & \frac{5}{3} \\
\hline
0 & & 1 & & 2 & & 3 & & 4 & & 5
\end{array}
$$

Example: Eigensteps of 3×5 UNTFs

$$
\begin{array}{ccccccccccc}
0 & \leq & 0 & \leq & \lambda_{3,2} & \leq & \lambda_{3,3} & \leq & \lambda_{3,4} & \leq & \frac{5}{3} \\
 & \rotatebox{45}{$\leq$} & & \rotatebox{45}{$\leq$} & & \rotatebox{45}{$\leq$} & & \rotatebox{45}{$\leq$} & & \rotatebox{45}{$\leq$} & \\
0 & \leq & 0 & \leq & \lambda_{2,2} & \leq & \lambda_{2,3} & \leq & \lambda_{2,4} & \leq & \frac{5}{3} \\
 & \rotatebox{45}{$\leq$} & & \rotatebox{45}{$\leq$} & & \rotatebox{45}{$\leq$} & & \rotatebox{45}{$\leq$} & & \rotatebox{45}{$\leq$} & \\
0 & \leq & \lambda_{1,1} & \leq & \lambda_{1,2} & \leq & \lambda_{1,3} & \leq & \lambda_{1,4} & \leq & \frac{5}{3} \\
\hline
0 & & 1 & & 2 & & 3 & & 4 & & 5
\end{array}
$$

Example: Eigensteps of 3×5 UNTFs

$$
\begin{array}{ccccccccccc}
0 & \leq & 0 & \leq & 0 & \leq & \lambda_{3,3} & \leq & \lambda_{3,4} & \leq & \frac{5}{3} \\
 & \nearrow\!\!\!/ & & \nearrow\!\!\!/ & & \nearrow\!\!\!/ & & \nearrow\!\!\!/ & & \nearrow\!\!\!/ & \\
0 & \leq & 0 & \leq & \lambda_{2,2} & \leq & \lambda_{2,3} & \leq & \lambda_{2,4} & \leq & \frac{5}{3} \\
 & \nearrow\!\!\!/ & & \nearrow\!\!\!/ & & \nearrow\!\!\!/ & & \nearrow\!\!\!/ & & \nearrow\!\!\!/ & \\
0 & \leq & 1 & \leq & \lambda_{1,2} & \leq & \lambda_{1,3} & \leq & \lambda_{1,4} & \leq & \frac{5}{3} \\
\hline
0 & & 1 & & 2 & & 3 & & 4 & & 5
\end{array}
$$

Example: Eigensteps of 3×5 UNTFs

$$
\begin{array}{ccccccccccc}
0 & \leq & 0 & \leq & 0 & \leq & \lambda_{3,3} & \leq & \lambda_{3,4} & \leq & \frac{5}{3} \\
 & \rotatebox{45}{$\leq$} & & \rotatebox{45}{$\leq$} & & \rotatebox{45}{$\leq$} & & \rotatebox{45}{$\leq$} & & \rotatebox{45}{$\leq$} & \\
0 & \leq & 0 & \leq & \lambda_{2,2} & \leq & \lambda_{2,3} & \leq & \frac{5}{3} & \leq & \frac{5}{3} \\
 & \rotatebox{45}{$\leq$} & & \rotatebox{45}{$\leq$} & & \rotatebox{45}{$\leq$} & & \rotatebox{45}{$\leq$} & & \rotatebox{45}{$\leq$} & \\
0 & \leq & 1 & \leq & \lambda_{1,2} & \leq & \lambda_{1,3} & \leq & \frac{5}{3} & \leq & \frac{5}{3} \\
\hline
0 & & 1 & & 2 & & 3 & & 4 & & 5
\end{array}
$$

Example: Eigensteps of 3×5 UNTFs

$$
\begin{array}{ccccccccccc}
0 & \leq & 0 & \leq & 0 & \leq & \lambda_{3,3} & \leq & \frac{2}{3} & \leq & \frac{5}{3} \\
 & \rotatebox{45}{$\leq$} & & \rotatebox{45}{$\leq$} & & \rotatebox{45}{$\leq$} & & \rotatebox{45}{$\leq$} & & \rotatebox{45}{$\leq$} & \\
0 & \leq & 0 & \leq & \lambda_{2,2} & \leq & \lambda_{2,3} & \leq & \frac{5}{3} & \leq & \frac{5}{3} \\
 & \rotatebox{45}{$\leq$} & & \rotatebox{45}{$\leq$} & & \rotatebox{45}{$\leq$} & & \rotatebox{45}{$\leq$} & & \rotatebox{45}{$\leq$} & \\
0 & \leq & 1 & \leq & \lambda_{1,2} & \leq & \frac{5}{3} & \leq & \frac{5}{3} & \leq & \frac{5}{3} \\
\hline
0 & & 1 & & 2 & & 3 & & 4 & & 5
\end{array}
$$

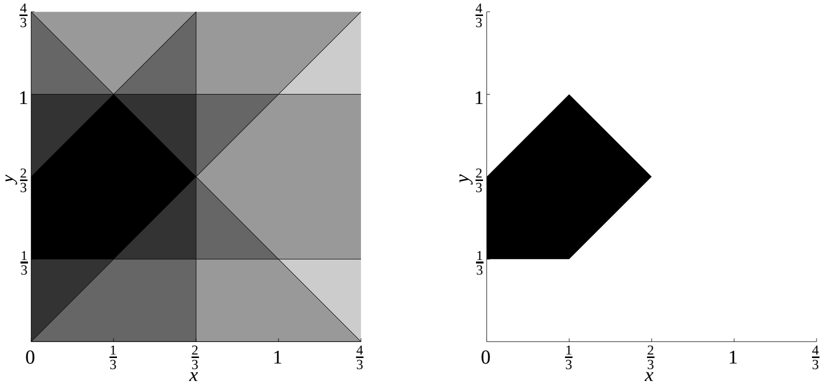
Example: Eigensteps of 3×5 UNTFs

0	$\leq$	0	$\leq$	0	$\leq$	x	$\leq$	$\frac{2}{3}$	$\leq$	$\frac{5}{3}$
	$\leq$		$\leq$		$\leq$		$\leq$		$\leq$	
0	$\leq$	0	$\leq$	y	$\leq$	$\lambda_{2,3}$	$\leq$	$\frac{5}{3}$	$\leq$	$\frac{5}{3}$
	$\leq$		$\leq$		$\leq$		$\leq$		$\leq$	
0	$\leq$	1	$\leq$	$\lambda_{1,2}$	$\leq$	$\frac{5}{3}$	$\leq$	$\frac{5}{3}$	$\leq$	$\frac{5}{3}$
0		1		2		3		4		5

Example: Eigensteps of 3×5 UNTFs

$$
\begin{array}{ccccccccccc}
0 & \leq & 0 & \leq & 0 & \leq & x & \leq & \frac{2}{3} & \leq & \frac{5}{3} \\
 & \rotatebox{45}{$\leq$} & & \rotatebox{45}{$\leq$} & & \rotatebox{45}{$\leq$} & & \rotatebox{45}{$\leq$} & & \rotatebox{45}{$\leq$} & \\
0 & \leq & 0 & \leq & y & \leq & \frac{4}{3} - x & \leq & \frac{5}{3} & \leq & \frac{5}{3} \\
 & \rotatebox{45}{$\leq$} & & \rotatebox{45}{$\leq$} & & \rotatebox{45}{$\leq$} & & \rotatebox{45}{$\leq$} & & \rotatebox{45}{$\leq$} & \\
0 & \leq & 1 & \leq & 2 - y & \leq & \frac{5}{3} & \leq & \frac{5}{3} & \leq & \frac{5}{3} \\
\hline
0 & & 1 & & 2 & & 3 & & 4 & & 5
\end{array}
$$

Example: Eigensteps of 3×5 UNTFs



(a) (b)

Eigensteps Polytope

Note: The set of all eigensteps arising from $M \times N$ UNTFs forms a convex polytope. [F, Mixon, Strawn & Poteet 13] gives an algorithm for constructing a particular type of “extreme” eigensteps which correspond to one of the corner points of this polytope.

Open Problems:

- How many corner points does this polytope have in general?
- What “strategies” do each of these corner points correspond to?
- What special properties do “corner point” UNTFs have?
- Can we use eigensteps to solve the Paulsen problem?

Disclaimer: I have only briefly read the paper Tim Haga and Christoph Pegel posted to arXiv on July 15. (and will present on Thursday?)

Weaver's ~~Conjecture~~ Theorem

- In 2013, Marcus, Spielman & Srivastava proved the famous **Kadison-Singer conjecture** by using the probabilistic method to prove the stronger **Weaver's conjecture:**

  There exists universal constants $\alpha > 2$ and $\beta > 0$ so that if $\{\boldsymbol{\varphi}_n\}_{n=1}^N$ is *any* α-tight frame for $\mathbb{F}^M$ where $\|\boldsymbol{\varphi}_n\| \leq 1$ for all n, then the frame elements can be partitioned into two frames $\{\boldsymbol{\varphi}_n\}_{n\in\mathcal{N}_1}$ and $\{\boldsymbol{\varphi}_n\}_{n\in\mathcal{N}_2}$ whose frame bounds lie between β and $\alpha - \beta$.

- In particular, they proved Weaver's conjecture holds for $\alpha = 18$ and $\beta = 2$, implying any UNTF of redundancy 18 can always be decomposed into two frames whose condition number is at most 8.

- **Open Problem:** How good of a partition can we compute deterministically (practically, numerically)?
  Is there a "square root bottleneck" á la deterministic RIP?

Part III:
Equiangular Tight Frames

Optimal Packings of Lines

Definition: The **coherence** of a set of unit vectors $\{\boldsymbol{\varphi}_n\}_{n=1}^N$ in $\mathbb{F}^M$ is

$$\max_{n\neq n'} |\boldsymbol{\varphi}_n^* \boldsymbol{\varphi}_{n'}|.$$

Frames of minimal coherence are called **Grassmannian frames**.

Note: Minimizing coherence is equivalent to packing lines: letting $\theta_{n,n'}$ denote the interior angle between the lines spanned by $\boldsymbol{\varphi}_n$ and $\boldsymbol{\varphi}_{n'}$,

$$\arg\min_{\{\boldsymbol{\varphi}_n\}} \Big(\max_{n\neq n'} |\boldsymbol{\varphi}_n^* \boldsymbol{\varphi}_{n'}|\Big) = \arg\max_{\{\boldsymbol{\varphi}_n\}} \Big(\min_{n\neq n'} \theta_{n,n'}\Big).$$

Welch Bound

Theorem: [Rankin 56, Welch 74, Strohmer & Heath 03]
The coherence of any unit vectors $\{\boldsymbol{\varphi}_n\}_{n=1}^N$ in $\mathbb{F}^M$ satisfies

$$\sqrt{\tfrac{N-M}{M(N-1)}} \leq \max_{n \neq n'} |\boldsymbol{\varphi}_n^* \boldsymbol{\varphi}_{n'}|,$$

with equality $\Leftrightarrow$ $\{\boldsymbol{\varphi}_n\}_{n=1}^N$ is an **equiangular tight frame (ETF)** for $\mathbb{F}^M$:

- $\{\boldsymbol{\varphi}_n\}_{n=1}^N$ is a UNTF and
- the modulus of inner products of distinct φ_n's is constant, i.e.

$$|(\boldsymbol{\Phi}^* \boldsymbol{\Phi})(n, n')| = |\boldsymbol{\varphi}_n^* \boldsymbol{\varphi}_{n'}| = \begin{cases} 1, & n = n', \\ \beta, & n \neq n'. \end{cases}$$

$$\textit{Proof:}\ \tfrac{N(N-M)}{M} \leq \sum_{n=1}^{N} \sum_{\substack{n'=1 \\ n' \neq n}}^{N} |\boldsymbol{\varphi}_n^* \boldsymbol{\varphi}_{n'}|^2 \leq N(N-1) \max_{n \neq n'} |\boldsymbol{\varphi}_n^* \boldsymbol{\varphi}_{n'}|^2.$$

Example: Optimally Packing 16 Lines In $\mathbb{R}^6$

$$\mathbf{\Phi} = \frac{1}{\sqrt{6}} \begin{bmatrix} + & + & + & + & + & + & + & + & + & + & + & + & + & + & + & + \\ + & - & + & - & + & - & + & - & + & - & + & - & + & - & + & - \\ + & - & - & + & + & - & - & + & + & - & - & + & + & - & - & + \\ + & + & + & + & - & - & - & - & + & + & + & + & - & - & - & - \\ + & - & + & - & + & - & + & - & - & + & - & + & - & + & - & + \\ + & - & - & + & - & + & + & - & - & + & + & - & + & - & - & + \end{bmatrix}$$

Example: Optimally Packing 16 Lines In $\mathbb{R}^6$

$$\mathbf{\Phi}\mathbf{\Phi}^* = \frac{16}{6}\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

Example: Optimally Packing 16 Lines In $\mathbb{R}^6$

$$\mathbf{\Phi}^*\mathbf{\Phi} = \frac{1}{3}\begin{bmatrix}
3 & - & + & + & + & - & + & - & + & + & + & + & + & - & - & + \\
- & 3 & + & + & - & + & - & + & + & + & + & + & - & + & + & - \\
+ & + & 3 & - & + & - & + & - & + & + & + & + & - & + & + & - \\
+ & + & - & 3 & - & + & - & + & + & + & + & + & + & - & - & + \\
+ & - & + & - & 3 & - & + & + & + & - & - & + & + & + & + & + \\
- & + & - & + & - & 3 & + & + & - & + & + & - & + & + & + & + \\
+ & - & + & - & + & + & 3 & - & - & + & + & - & + & + & + & + \\
- & + & - & + & + & + & - & 3 & + & - & - & + & + & + & + & + \\
+ & + & + & + & + & - & - & + & 3 & - & + & + & + & - & + & - \\
+ & + & + & + & - & + & + & - & - & 3 & + & + & - & + & - & + \\
+ & + & + & + & - & + & + & - & + & + & 3 & - & + & - & + & - \\
+ & + & + & + & + & - & - & + & + & + & - & 3 & - & + & - & + \\
+ & - & - & + & + & + & + & + & + & - & + & - & 3 & - & + & + \\
- & + & + & - & + & + & + & + & - & + & - & + & - & 3 & + & + \\
- & + & + & - & + & + & + & + & + & - & + & - & + & + & 3 & - \\
+ & - & - & + & + & + & + & + & - & + & - & + & + & + & - & 3
\end{bmatrix}$$

The Grassmannian Frame Problem

- For any $M \leq N$, we want Grassmannian frames. If there exists an $M \times N$ ETF, it is Grassmannian.
- However, for many most choices of M and N, we do not know whether an $M \times N$ ETF exists. Moreover, for many choices of M and N, we know that $M \times N$ ETF cannot exist.
- Almost all research in this area has used the following program:
  - Find as many explicit constructions of ETFs as possible.
  - Find the strongest possible necessary conditions on ETF existence.

Open Problem: Find Grassmannian ETFs for cases of M and N for which no ETF exists. In particular, find ways of proving that $\{\varphi_n\}_{n=1}^{N}$ has optimal coherence that do not involve equiangularity.

Some Known Constructions of ETFs

Fact: All known infinite families of ETFs involve some type of combinatorial design, including:

- **harmonic ETFs** arising from **difference sets** in abelian groups, e.g. $\{(0,0,0,0),(0,0,1,0),(1,0,0,0),(1,0,0,1),(1,1,0,0),(1,1,1,1)\}$ regarded as a subset of $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$:

$-$	$(0,0,0,0)$	$(0,0,1,0)$	$(1,0,0,0)$	$(1,0,0,1)$	$(1,1,0,0)$	$(1,1,1,1)$
$(0,0,0,0)$	$(0,0,0,0)$	$(0,0,1,0)$	$(1,0,0,0)$	$(1,0,0,1)$	$(1,1,0,0)$	$(1,1,1,1)$
$(0,0,1,0)$	$(0,0,1,0)$	$(0,0,0,0)$	$(1,0,1,0)$	$(1,0,1,1)$	$(1,1,1,0)$	$(1,1,0,1)$
$(1,0,0,0)$	$(1,0,0,0)$	$(1,0,1,0)$	$(0,0,0,0)$	$(0,0,0,1)$	$(0,1,0,0)$	$(0,1,1,1)$
$(1,0,0,1)$	$(1,0,0,1)$	$(1,0,1,1)$	$(0,0,0,1)$	$(0,0,0,0)$	$(0,1,0,1)$	$(0,1,1,0)$
$(1,1,0,0)$	$(1,1,0,0)$	$(1,1,1,0)$	$(0,1,0,0)$	$(0,1,0,1)$	$(0,0,0,0)$	$(0,0,1,1)$
$(1,1,1,1)$	$(1,1,1,1)$	$(1,1,0,1)$	$(0,1,1,1)$	$(0,1,1,0)$	$(0,0,1,1)$	$(0,0,0,0)$

Singer and **McFarland** difference sets give harmonic ETFs of size

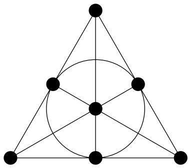
$$\left(\frac{q^{j+1}-1}{q-1}\right)\times\left(\frac{q^{j+2}-1}{q-1}\right), \quad q^j\left(\frac{q^{j+1}-1}{q-1}\right)\times q^{j+1}\left(\frac{q^{j+1}-1}{q-1}+1\right),$$

respectively, for any prime power q and any positive integer j.

Some Known Constructions of ETFs

Fact: All known infinite families of ETFs involve some type of combinatorial design, including:

- **Steiner ETFs** from **balanced incomplete block designs** e.g.



Fano plane

Some Known Constructions of ETFs

Fact: All known infinite families of ETFs involve some type of combinatorial design, including:

- **Steiner ETFs** from **balanced incomplete block designs** e.g.

$$\begin{bmatrix} + & + & 0 & + & 0 & 0 & 0 \\ 0 & + & + & 0 & + & 0 & 0 \\ 0 & 0 & + & + & 0 & + & 0 \\ 0 & 0 & 0 & + & + & 0 & + \\ + & 0 & 0 & 0 & + & + & 0 \\ 0 & + & 0 & 0 & 0 & + & + \\ + & 0 & + & 0 & 0 & 0 & + \end{bmatrix}$$

Incidence matrix of the corresponding **Steiner triple system**

Some Known Constructions of ETFs

Fact: All known infinite families of ETFs involve some type of combinatorial design, including:

- **Steiner ETFs** from **balanced incomplete block designs** e.g.

$$\begin{bmatrix} + & + & 0 & + & 0 & 0 & 0 \\ 0 & + & + & 0 & + & 0 & 0 \\ 0 & 0 & + & + & 0 & + & 0 \\ 0 & 0 & 0 & + & + & 0 & + \\ + & 0 & 0 & 0 & + & + & 0 \\ 0 & + & 0 & 0 & 0 & + & + \\ + & 0 & + & 0 & 0 & 0 & + \end{bmatrix} \text{“}\otimes\text{”} \begin{bmatrix} + & - & + & - \\ + & + & - & - \\ + & - & - & + \end{bmatrix}$$

$$= \left[\begin{array}{cccc|cccc|cccc|cccc|cccc|cccc|cccc} + & - & + & - & + & + & - & - & 0 & 0 & 0 & 0 & + & - & - & + & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & + & - & + & - & + & + & - & - & 0 & 0 & 0 & 0 & + & - & - & + & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & + & - & + & - & + & + & - & - & 0 & 0 & 0 & 0 & + & - & - & + & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & + & - & + & - & + & + & - & - & 0 & 0 & 0 & 0 & + & - & - & + \\ + & - & - & + & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & + & - & + & - & + & + & - & - & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & + & - & - & + & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & + & - & + & - & + & + & - & - \\ + & + & - & - & 0 & 0 & 0 & 0 & + & - & - & + & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & + & - & + & - \end{array}\right]$$

"Inflating" the Steiner system by a regular simplex to form an ETF

Some Known Constructions of ETFs

Fact: All known infinite families of ETFs involve some type of combinatorial design, including:

- **Steiner ETFs** from **balanced incomplete block designs** e.g.

  ETFs of size:
  - $\left(\frac{q^{j+1}-1}{q-1}\right) \times \left(\frac{q^{j+2}-1}{q-1}\right)$ from **affine geometries**,
  - $\frac{(q^{j+1}-1)(q^{j+2}-1)}{(q+1)(q-1)^2} \times \frac{q^{j+2}-1}{q-1}(1 + \frac{q^{j+1}-1}{q-1})$ from **projective geometries**,
  - $\frac{(2^s+1)(2^{r+s}+2^r-2^s)}{2^r} \times (2^s + 2)(2^{r+s} + 2^r - 2^s)$ from **Denniston designs**,

  for any prime power q and any positive integers j, $2 \leq r < s$.

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Finding new explicit constructions of ETFs seems really hard and is probably not well-suited to the "large collaborative group" setting of workshops... maybe we should focus on necessary conditions instead?

Absolute Bounds

Theorem: [Gerzon in Lemmens & Seidel 73]
If unit vectors $\{\varphi_n\}_{n=1}^N$ are equiangular and not collinear in $\mathbb{F}^M$ then

$$N \leq \binom{M+1}{2} \text{ if } \mathbb{F} = \mathbb{R}, \qquad N \leq M^2 \text{ if } \mathbb{F} = \mathbb{C}.$$

If these bounds are achieved then $\{\varphi_n\}_{n=1}^N$ is necessarily an ETF for $\mathbb{F}^M$.

Proof Sketch: $\{\varphi_n\}_{n=1}^N$ being equiangular but not collinear implies their projection operators $\{\varphi_n \varphi_n^*\}_{n=1}^N$ are linearly independent.

Note: When $\mathbb{F} = \mathbb{R}$, $N = \binom{M+1}{2}$ is known to not be achievable for many M due to integrality conditions given on the next slide.
It is achievable for $M = 3, 7, 23$.

When $\mathbb{F} = \mathbb{C}$, $N = M^2$ is known to be achievable for many M, and is conjectured to be always so (see Dustin's talk tomorrow!)

Integrality Conditions

Theorem: [Sustik, Tropp, Dhillon & Heath 07]
If a **real** $M \times N$ ETF exists and $1 < M < N - 1$ with $N \neq 2M$, then

$$\left[\frac{M(N-1)}{N-M}\right]^{\frac{1}{2}}, \quad \left[\frac{(N-M)(N-1)}{N-1}\right]^{\frac{1}{2}}$$

are necessarily odd integers.

Proof Sketch: Study the eigenvalues of the matrix obtained by converting $\mathbf{\Phi}^*\mathbf{\Phi}$ into a $\{-1, 0, 1\}$-valued matrix.

This is closely related to a well known equivalence between real ETFs and **strongly regular graphs**.

Note: These necessary conditions are not sufficient, e.g. 47×1128.

Complex Integrality Conditions?

- In the complex case, the only necessary conditions we have on ETFs is the absolute bound on it and its **Naimark complement**

$$N \leq M^2, \quad N \leq (N-M)^2.$$

- Well, that's not quite true... in 2014, Ferenc Szöllősi used algebraic geometry to prove that there does not exist a 3×8 complex ETF!

- Based on this "overwhelming" evidence (and a lot of explicit constructions of complex ETFs, and my suspicion that it will be extremely hard to prove it either true or false), I conjecture the following:

  Conjecture: If there exists a complex $M \times N$ ETF, then one of the three integers M, $N-1$ and $N-M$ must divide the product of the other two.

Summary

- A lot of finite frame theory is about generalizing orthonormal bases.
- Tight frames generalize the Pythagorean theorem, are commonplace and (barring additional restrictions) are easy to construct.
- Unit norm tight frames (UNTFs) are much harder to construct.
  - Nevertheless, UNTFs exist for every $M \leq N$.
  - The fact that the Paulsen problem is open tells us we still don't really understand the geometry of the set of all $M \times N$ UNTFs.
  - Eigensteps are a way of parametrizing this set, and raise their own questions.
- Equiangular tight frames are even more rare.
  - In the complex case, we have almost no necessary conditions on their existence.
  - For M and N for which no ETF exists, we have almost no techniques for proving given vectors are Grassmannian.