# Homology cycle bases from acyclic matchings

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What is . . .

# Applied Topology ?

- studying global features of shapes
- applications in other branches of mathematics
- applications in computation and in other sciences

## Combinatorial cell complexes

**locally** = simple cell attachments

for example - simplicial or prodsimplicial complexes;

**globally** = cells are indexed by combinatorial objects

such as graphs, partitions, permutations, various combinations and enrichments of these;

**taking boundary** = combinatorial rule for the indexing objects

for example - removing vertices in graphs, merging blocks in partitions, relabeling, etc.

## Combinatorial means

using matchings, orderings, labelings, et cetera, to simplify or to completely eliminate algebraic computations and topological deformations.

A very useful tool is provided by the notion of simplicial collapses.

**Definition.**

An **elementary simplicial collapse** in a simplicial complex $K$ is the removal of a pair of simplices $(\sigma, \tau)$, such that

- $\tau$ is a maximal simplex;
- $\dim \sigma = \dim \tau - 1$;
- $\tau$ is the only maximal simplex containing $\sigma$.

A simplicial complex $K$ is called **collapsible** if there exists a sequence of elementary simplicial collapses reducing $K$ to a vertex.

## The idea of Discrete Morse Theory.

The homotopy type is clear if we can remove some top-dimensional simplices and obtain a collapsible complex.

More generally, we may want to make **internal collapses** and trace how the remaining **critical** simplices are glued on afterwards.

**Definition.**

Assume we are given a simplicial complex $K$, and a partial matching $\mu : W \to W$ on the set of simplices of $K$.

This matching is called **acyclic**, if there does not exist a cycle of the following form:

$$b_1 \succ \mu(b_1) \prec b_2 \succ \mu(b_2) \prec \cdots \prec b_n \succ \mu(b_n) \prec b_1,$$

with $n \geq 2$, and all $b_i \in W$ being distinct.

**Theorem.**

Assume $K$ is an abstract simplicial complex, and assume $K'$ is a simplicial subcomplex of $K$. The following statements are equivalent:

(1) there exists a sequence of elementary collapses leading from $K$ to $K'$;

(2) there exists an acyclic matching on the set of the simplices of $K$ which are not contained in $K'$.

**Theorem.**

Assume $K$ is a simplicial complex, $M$ is some set of simplices of $K$, and $\mu : M \to M$ is an acyclic matching. Then there exists a CW complex $X$ such that

(1) for each dimension $d$, the number of $d$-cells in $X$ is equal to the number of $d$-simplices of $K$, which do not belong to $M$.

(2) we have a homotopy equivalence $K \simeq X$.

The simplices of $K$ which do not belong to $M$ are called **critical** with respect to $\mu$.

This theorem is a very important result since it allows us to replace $K$ with a potentially much smaller CW complex.

**Definition.**

Assume we are given two arbitrary posets $P$ and $Q$, and a poset map $\varphi : P \to Q$. The map $\varphi$ is called a **poset map with small fibers**, if for any $q \in Q$, one of the following three statements is true:

- the fiber $\varphi^{-1}(q)$ is empty;
- the fiber $\varphi^{-1}(q)$ consists of a single element;
- the fiber $\varphi^{-1}(q)$ consists of two comparable elements.

In particular, we have a partial matching $M(\varphi)$.

**Theorem: Acyclic matchings via poset maps with small fibers.**

The following two statements describe the close relation between poset maps with small fibers and acyclic matchings.

(1) For any poset map with small fibers $\varphi : P \to Q$, the partial matching $M(\varphi)$ is acyclic.

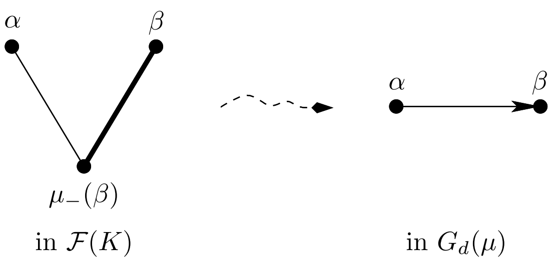
(2) Any acyclic matching on $P$ can be represented as $M(\varphi)$ for some poset map with small fibers $\varphi$.

More in the upcoming book on **Discrete Morse theory.**

**Definition.**

Assume we are given an abstract simplicial complex $K$, and an acyclic matching $\mu$. Pick $0 \leq d \leq \dim K$. The oriented graph $G_d(\mu)$ is defined as follows:

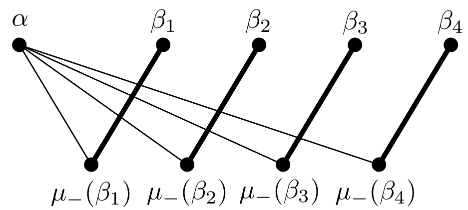
- the vertices of $G_d(\mu)$ are indexed by the $d$-dimensional simplices of $K$;
- the edges of $G_d(\mu)$ are given by the rule: $(\alpha, \beta)$ is an edge of $G_d(\mu)$ if and only if $\mu_-(\beta)$ is defined, and $\alpha \succ \mu_-(\beta)$.




The rule defining the edges of the oriented graph $G_d(\mu)$.

The recursive rule for $\varphi$

$$\varphi(\alpha) := \alpha + \sum_{i=1}^{m} \varphi(\beta_i)$$




**Theorem.** Values of $\varphi$ on critical simplices give a homology basis.

A source of combinatorial complexes is provided by complexes associated to monotone graph properties.

**Definition.**

Assume $n$ is a natural number. A **graph property** $\Gamma$ is a collection of isomorphism classes of graphs on the vertex set $\{1, \ldots, n\}$.

A graph property is called **monotone** if: whenever a graph $G$ has property $\Gamma$, any graph obtained from $G$ by the deletion of some of the edges will also have the property $\Gamma$.

Given a non-trivial monotone graph property $\Gamma$, we define the simplicial complex $X(\Gamma)$ as follows:

- the vertices of $X(\Gamma)$ are all potential edges, that is, all pairs $(i, j)$, $1 \leq i < j \leq n$;
- the set of vertices forms a simplex of $X(\Gamma)$ if the corresponding graph has the property $\Gamma$.

For example, one can consider the **simplicial complex of all disconnected graphs** (but not of all connected graphs!).

Cycles for the complex of disconnected graphs.

Standard acyclic labeling is given iteratively by first trying $\sigma$ XOR $(1,2)$, then $\sigma$ XOR $(i,3)$ etc.

$\varphi(\sigma) = \partial(T_\sigma)$, where $T_\sigma$ is a *recursive tree.*

**Definition.**

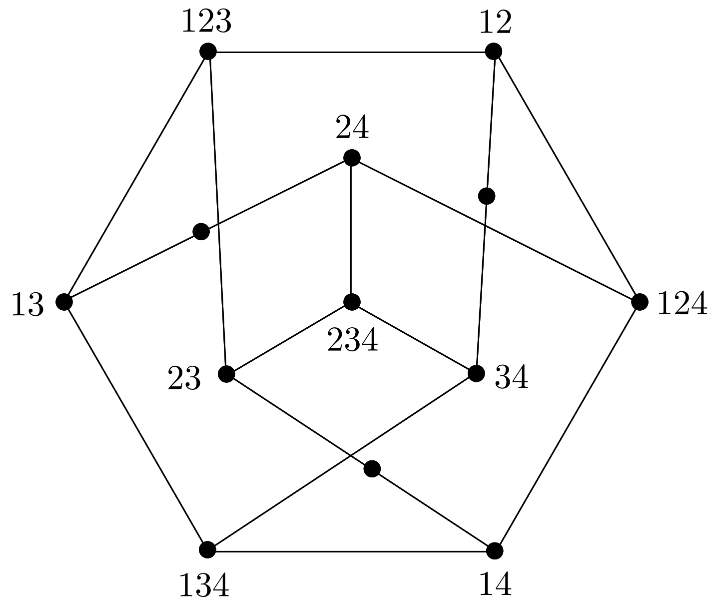
Given a positive integer $n$, the **partition lattice** $\Pi_n$ consists of all proper partitions of the set $\{1, \ldots, n\}$, ordered by refinement.

We see that:

- the vertices of the simplicial complex $\Delta(\Pi_n)$ are indexed by all proper partitions of the set $\{1, \ldots, n\}$,
- the simplices of $\Delta(\Pi_n)$ are all refinement chains of partitions.

In particular, the simplicial complex $\Delta(\Pi_n)$ is **pure**, meaning all its top-dimensional simplices have the same dimension, and it has dimension $n - 3$.

For example, for $n = 4$ we get the following simplicial complex.



**Theorem.**

The simplicial complex $\Delta(\Pi_n)$ is homotopy equivalent to a wedge of $(n-1)!$ spheres of dimension $n-3$.

So, for example,

- $\Delta(\Pi_4)$ is homotopy equivalent to a wedge of 6 spheres of dimension 1,
- $\Delta(\Pi_5)$ is homotopy equivalent to a wedge of 24 spheres of dimension 2, etc.

Cycles for the partition lattice.

Any chain $\sigma$ in $\Delta(\Pi_n)$ can be written as

$$(\alpha_1 n) < \cdots < (\alpha_1 \dots \alpha_k n) < \pi < \dots$$

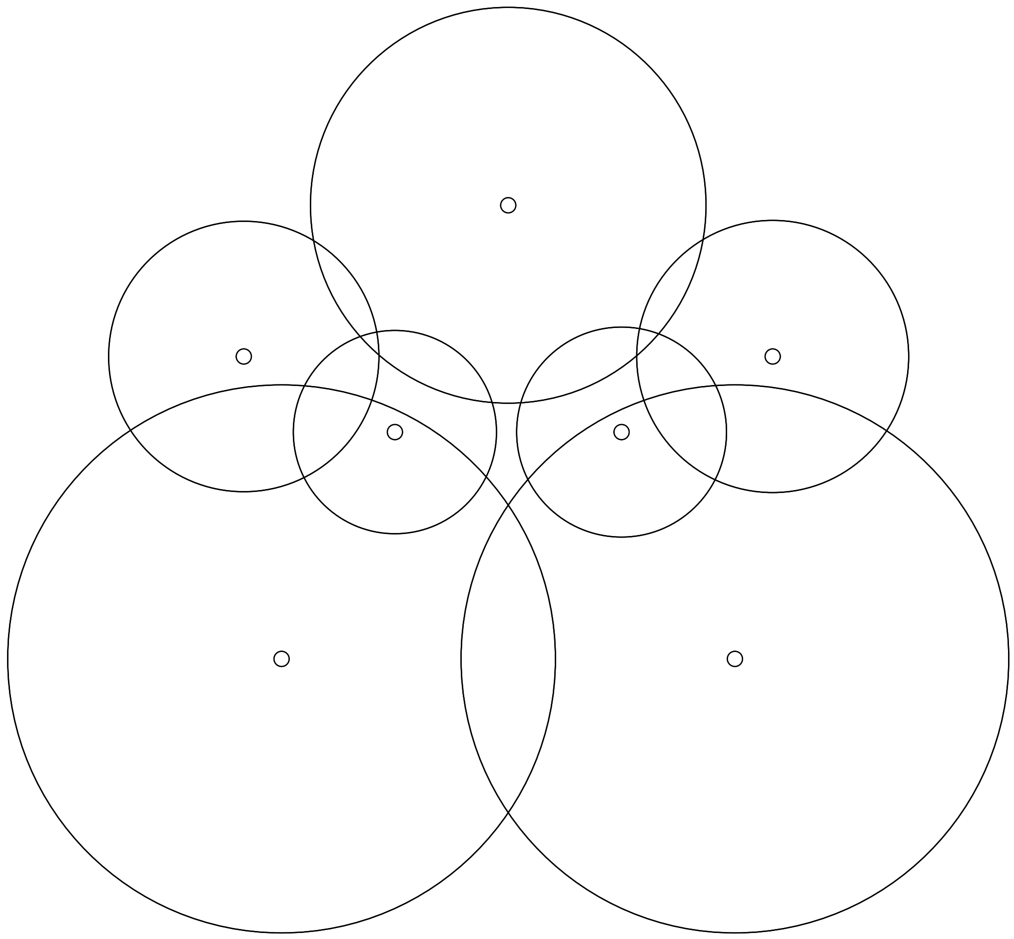
where $\pi$ is not of the form $(\alpha_1 \dots \alpha_k \alpha_{k+1} n)$.

Set $B := \{\alpha_1 \dots \alpha_k n\}$. The *pivot* is the partition obtained by *chopping off* $B$ in the first possible partition in $\sigma$.

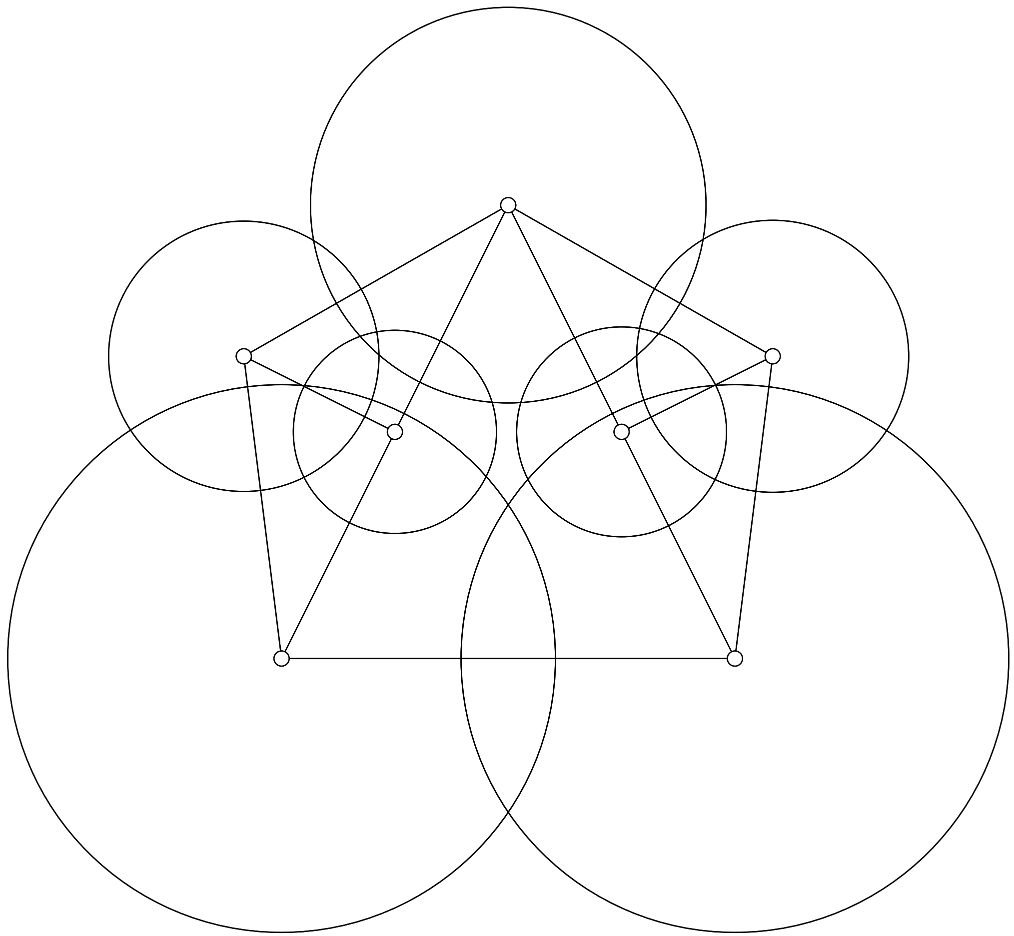
Set $\mu(\sigma) := \mu(\sigma)$ XOR pivot.

Our technique finds the classes associated to imbedded Boolean algebras.

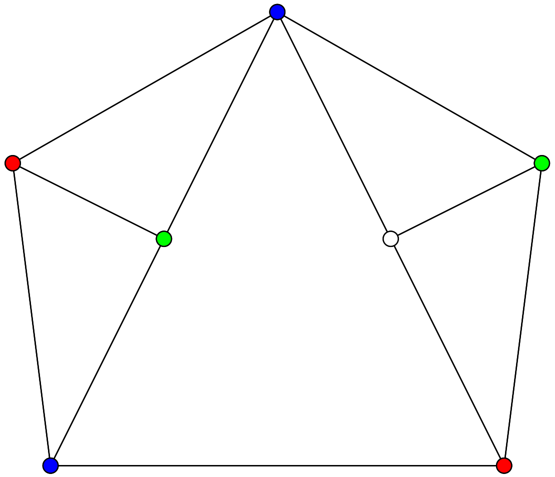
## Graph colorings



## Graph colorings



## Graph colorings



**Chromatic number** of a graph $G$, denoted $\chi(G)$, is the smallest number of colors needed to color the vertices of G so that no two adjacent vertices share the same color.

Calculating the chromatic number of a graph is an NP-complete problem.

## The topology of graph colorings

### Ansatz

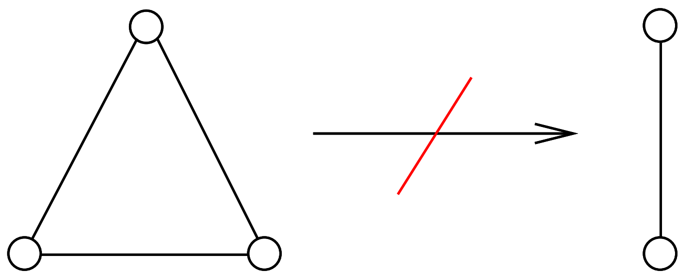
$$\begin{array}{ccc} \text{graph } G & \longrightarrow & \text{topological space } X(G) \\ & & \downarrow \\ \begin{array}{c}\text{combinatorial} \\ \text{properties of } G\end{array} & \longleftarrow & \begin{array}{c}\text{topological} \\ \text{properties of } X(G)\end{array} \end{array}$$

# The topology of graph colorings

**Definition.**

A **graph homomorphism** between graphs $T$ and $G$ is a map $\varphi : V(T) \to V(G)$, such that for every edge $(x, y)$ in $T$ the pair $(\varphi(x), \varphi(y))$ is an edge in $G$.

**Observation:** $G$ is $n$-colorable $\Leftrightarrow$ $\exists\, \varphi : G \to K_n$



The composition of two homomorphisms $\varphi_1 : G_1 \to G_2$ and $\varphi_2 : G_2 \to G_3$ is again a homomorphism $\varphi_2 \circ \varphi_1 : G_1 \to G_3$.

We obtain the category **Graphs** with graphs as objects and graph homomorphisms as morphisms.

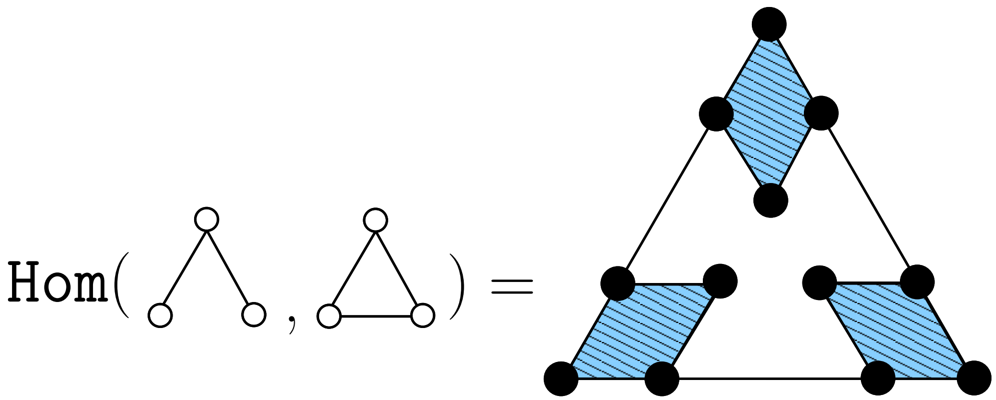
## The topology of graph colorings

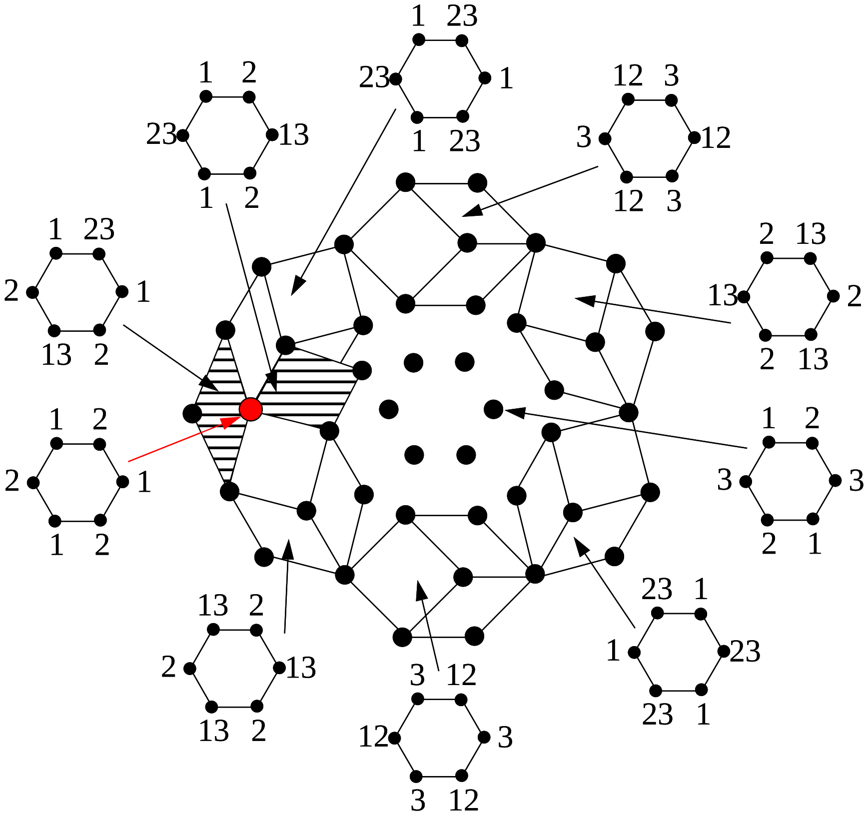
A cell in $\texttt{Hom}\,(T, K_n)$ is an assignment of subsets of $[n]$ to vertices of $T$, such that an arbitrary choice of one color per list yields an admissible coloring of $T$.

Now replace $K_n$ with an arbitrary graph $G$.

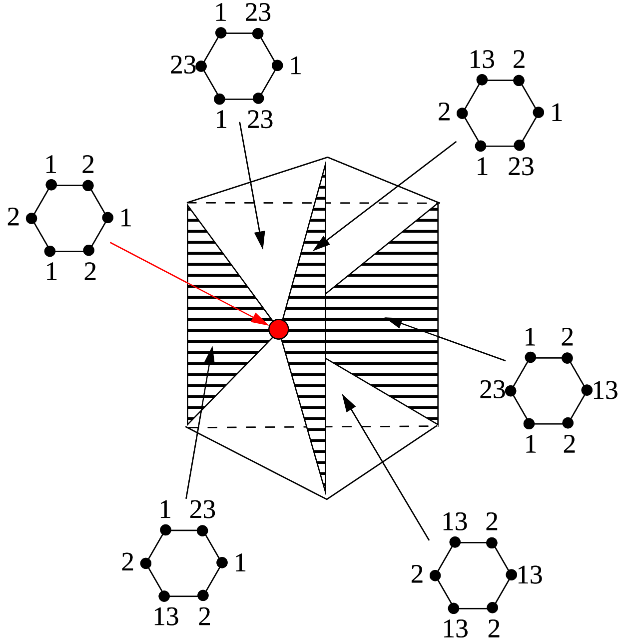
- The vertices of $G$ are the colors.
- Homomorphisms $T \to G$ replace the valid colorings.

**Example.**




1 23
23 1
1 23
1 2
23 13
1 2
12 3
3 12
12 3
1 23
2 1
13 2
2 13
13 2
2 13
1 2
2 1
1 2
1 2
3 3
2 1
13 2
2 13
13 2
3 12
12 3
3 12
23 1
1 23
23 1

Hom $(C_6, K_3)$


1 23
23 1
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13 2
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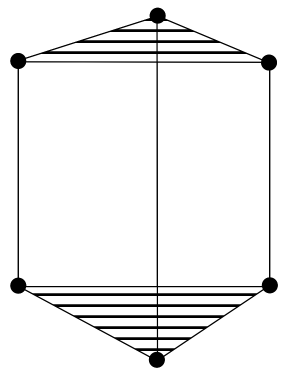
13 2



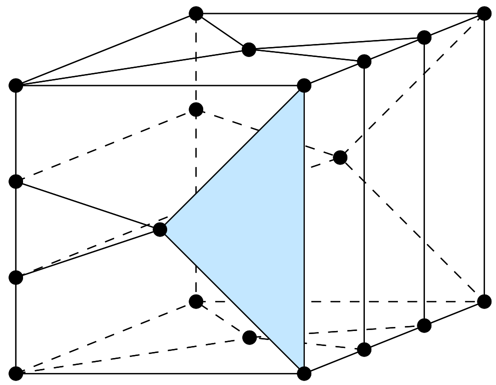
Consider the special case $\texttt{Hom}\,(K_m, K_n)$, where $K_m$ and $K_n$ are complete graphs on $m$, resp. $n$, vertices.

In that case the cells are indexed by all $m$-tuples $(A_1, \dots, A_m)$, such that $A_i$'s are non-empty disjoint subsets of $[n]$.

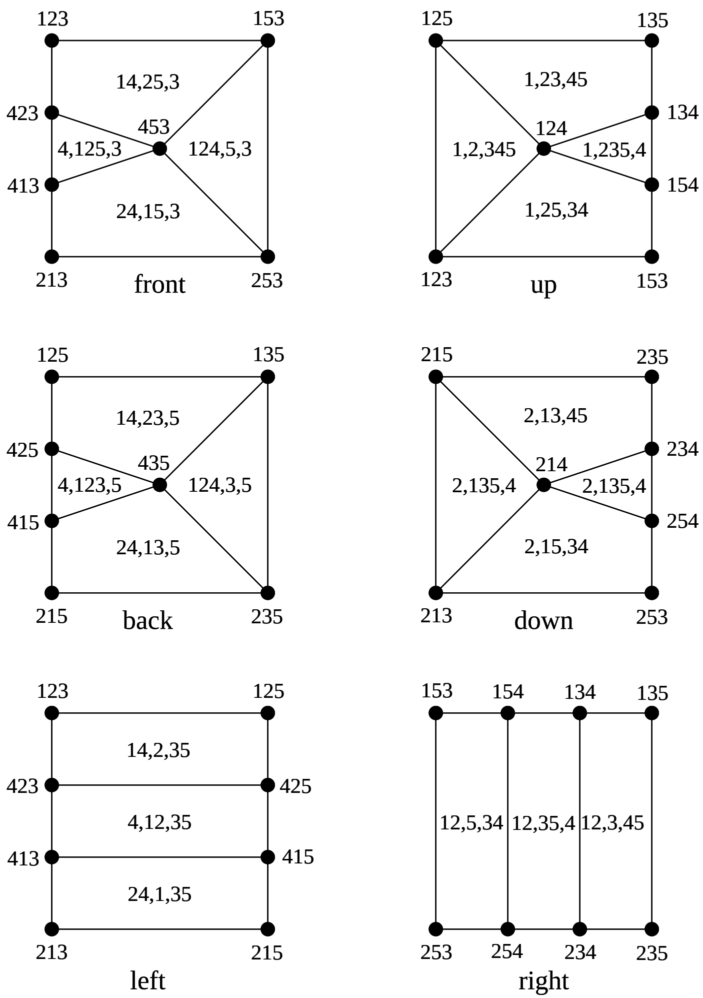
In particular, the vertices of $\texttt{Hom}\,(K_m, K_n)$ are simply all $m$-tuples $(v_1, \dots, v_m)$, such that $v_i \in [n]$, and $v_i \neq v_j$, while for a maximal-dimensional cell $(A_1, \dots, A_m)$, we have $A_1 \cup \dots \cup A_m = [n]$. The combinatorial rule for the boundary operation is the removal of elements from $A_i$'s.

Acyclic matching for $\texttt{Hom}\,(K_m, K_n)$ (Idea):

start scanning from the right and try to insert the maximal missing element so that it comes out on top of this set.



A cycle in $\texttt{Hom}\,(K_3, K_5)$.


123
153
14,25,3
423
453
4,125,3
124,5,3
413
24,15,3
213
front
253
125
135
1,23,45
124
134
1,2,345
1,235,4
154
1,25,34
123
up
153
125
135
14,23,5
425
435
4,123,5
124,3,5
415
24,13,5
215
back
235
215
235
2,13,45
214
234
2,135,4
2,135,4
254
2,15,34
213
down
253
123
125
14,2,35
423
425
4,12,35
413
415
24,1,35
213
215
left
153
154
134
135
12,5,34
12,35,4
12,3,45
253
254
234
235
right

**Algorithm 1** The algorithm computing explicit homology cycles

```
for all d-cells v do
   set out(v) := 0 {Initialize the outdegrees}
   for all d-cells w do
      set G(v, w) := false {Initialize the graph}
   end for
end for
for all d-cells v do
   for all (d − 1)-cells u in ∂v do
      if u ∈ N_{d−1} then
         set G(v, μ(u)) := true {Create an edge}
         set out(v) := out(v) + 1 {Increase the outdegree}
      end if
   end for
end for
for all d-cells v do
   set φ(v) := v {Initialize the φ function}
end for
let S be the set of all v, such that out(v) = 0 {Put all the sinks in S}
repeat
   pick v ∈ S
   for vertices w, such that G(w, v) do
      set out(w) := out(w) − 1 {The edge (w, v) is spent}
      set φ(w) := φ(w) + φ(v) {Update the value of φ on w}
      if out(w) = 0 then
         add w to the set S {If w is a new sink, then add it to the list}
      end if
   end for
   remove v from S {The vertex v is spent}
until the set S is empty
```